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Comparative Numerical Analysis of Newton and Modified Newton's Law of Cooling

Nurul Nazirah Zulkifli[1], Norzuria Ibrahim[1]*

[1] *Department of Mathematics and Statistics, Faculty of Applied Sciences and Technology, UTHM Kampus Cawangan Pagoh, Hab Pendidikan Tinggi Pagoh, KM 1, Jalan Panchor, 84600 Pagoh, Muar, Johor, MALAYSIA.*

*Corresponding Author: norzuria@uthm.edu.my
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Abstract

The aim of this research was to comprehend how an object decays in temperature under different conditions by conducting a comparative numerical analysis of the classical Newton's Law of Cooling and a reformulated Newton's Law of Cooling. The main purpose of this study was to apply numerical solutions to two cooling models and compare their performance with the Fourth Order Runge-Kutta method (RK4) and Dormand-Prince method. Both models were solved analytically to be used as reference solutions in order to evaluate the accuracy of the numerical methods. The sensitivity and stability of the methods were tested by numerical simulation using various step sizes and absolute error analysis was done to measure the differences between numerical and analytical solutions. To show the cooling behaviour of the two models with time, graphical representations were used. The results showed that the RK4 and Dormand-Prince can be used to provide an accurate approximation of the analytical solutions, and that the Dormand-Prince method has a better prediction by the fact that it has the ability to control the step size. Furthermore, the modified Newton's Law of Cooling was a better representation of the observed colling patterns than the classical model, which pointed to its greater flexibility in predicting real colling behaviour. On the whole, this paper has shown that numerical techniques are effective in the formulation of the cooling problems and how the modified model was useful in the realistic representation of the thermal behaviour, which may be considered to be a systematic way of conducting a thermal analysis in the future in the context of engineering.

1. Introduction

The concept of temperature decay can be defined as the process whereby the temperature of an object decreases with time as it transfers heat to the immediate environment. Among the basic models of temperature decay is Newton's law of cooling. This law is commonly used in situations when the temperature of an object is coming close to that of its environment like the cooling of a hot cup of coffee in a room [1].

Although it is widely used, the classical Newton's Law of Cooling assumes simplification, which includes the assumption of even temperature distribution in the body and little difference in temperature between the object and the environment [2]. Such assumptions are untrue in all cases and particularly in the field of practical uses that make use of fluids, complex materials or experimental conditions where other mechanisms of heat transfer like convection and radiation become important. Consequently, there is a tendency to have inconsistencies between theoretical forecasts and experimental results.

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To overcome these shortcomings, Newton's Law of Cooling has been modified by considering new parameters as an effort to enhance performance of the model. A change, involving a non-linear time exponent in the cooling model, brings the decay of the temperature closer to the real-life experimental behaviour. This is a modification of the Newton law of cooling that has been discovered to provide greater comparison to experimental data than the classical form of the law of cooling [2].

Nonetheless, in spite of the variations, the two models are both aimed at explaining the dynamics of temperature decay in systems whereby the temperature is caused by external processes [3]. Differential equations characterize such models which in the case of complex systems can be highly challenging to solve explicitly. This is the reason why the approximate solutions are obtained using numerical methods [4].

Numerical methods such as Runge-Kutta (RK) method and Dormand-Prince method are frequently used to solve such differential equations. The RK methods refer to a family of methods of iteration which provide correct approximations of ordinary differential equations (ODEs). The fourth-order Runge-Kutta method (RK4) is the most popular and common of them all and is employed in the solution of temperature decay equations since it is a trade-off between easy and accuracy [5].

The Dormand-Prince method is another numerical method whose goal is to increase the accuracy of solutions at a lower cost of calculations. It is an embedded RK process that is highly accurate with special effectiveness where stiff problems are present or where a choice of adaptive steps sizes is required to guarantee greater precision [5].

This study relied on Python as a primary programming language to write and solve the Newton law of cooling and modified Newton law of cooling among the two numerical algorithms namely RK4 and Dormand-Prince.

2. Methodology

The research technique of this paper is to solve the classical and modified form of the Newton Law of Cooling by using the RK4 and Dormand-Prince methods. The two models are tested using RK4 and Dormand-Prince method with the two step sizes h = 0.1 and h = 0.05. The results were obtained in the study conducted by [6], both the results were based on the experiments and the precise solutions of the Newtons Law of Cooling. This data and temperatures of the various time intervals were used to calculate the cooling constant k and established the accuracy of the numerical processes involved in the current study. All calculations are performed using Python.

2.1 Mathematical Models

2.1.1 Newton's Law of Cooling

Newton's Law of Cooling explains the process of heat loss and the rate of transfer of heat from hot objects to the environment [1]. The law is expressed mathematically by the following equation [2]:

$$\frac{\mathrm{dT(t)}}{\mathrm{dt}} = \lambda(\mathrm{T(t)} - \mathrm{T_a}) \tag{1}$$

where:
$T(t)$ = the temperature of heated body at time t,
T_a = the ambient temperature,
λ = the constant of proportionality and is defined as negative represent the heat loss from the heated body. The sign of λ is changeable because some sources interpreted λ to be positive.

2.1.2 Modified Newton's Law of Cooling

The proposed modification version by [2] introducing a nonlinear exponent to the temperature difference term. This modification allows the cooling rate to vary nonlinearly with respect to the temperature difference between the object and its surroundings. As a result, the modified model provides greater flexibility in capturing experimental cooling data, especially in cases where the classical linear assumption is inadequate. In particular, we add a parameter ψ that is the time exponent to the differential equation in the following way [2]:

$$\frac{\mathrm{dT}}{\mathrm{dt}} = \lambda\psi t^{\psi-1}(T - T_a) \tag{2}$$

where:
ψ = a newly introduced parameter that modifies the time exponent in the cooling equation.

2.2 Numerical Methods

2.2.1 Fourth Order Runge-Kutta Method (RK4)

The RK methods are a family of iterative techniques used to solve ODEs. The RK4 is the most prevalent one and offers a reasonable trade off between accuracy and computational efficiency [7]. The RK4 method uses the following formula [8]:

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{3}$$

where:

$$k_1 = hf(x_i, y_i) \tag{4}$$

$$k_2 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right) \tag{5}$$

$$k_3 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right) \tag{6}$$

$$k_4 = hf(x_i + h, y_i + k_3) \tag{7}$$

2.2.2 Dormand-Prince Method

One of the versions of RK method is the Dormand-Prince method [5]. It employs both the 4th and 5th orders of the RK method to obtain the numerical solution, with the absolute difference between them representing the error in the solution [9]. The Dormand–Prince method employs an embedded Runge–Kutta pair to estimate the local truncation error. If the estimated error exceeds the prescribed tolerance, the step size is reduced; otherwise, the step size is increased to improve computational efficiency while maintaining accuracy.

$$k_1 = hf(x_i, y_i) \tag{8}$$

$$k_2 = hf\left(x_i + \frac{1}{5}h, y_i + \frac{1}{5}k_1\right) \tag{9}$$

$$k_3 = hf\left(x_i + \frac{3}{10}h, y_i + \frac{3}{40}k_1 + \frac{9}{40}k_2\right) \tag{10}$$

$$k_4 = hf\left(x_i + \frac{4}{5}h, y_i + \frac{44}{45}k_1 - \frac{56}{15}k_2 + \frac{32}{9}k_3\right) \tag{11}$$

$$k_5 = hf\left(x_i + \frac{8}{9}h, y_i + \frac{19372}{6561}k_1 - \frac{25360}{2187}k_2 + \frac{64448}{6561}k_3 - \frac{212}{729}k_4\right) \tag{12}$$

$$k_6 = hf\left(x_i + \frac{8}{9}h, y_i + h, y_i + \frac{9017}{3168}k_1 - \frac{355}{33}k_2 + \frac{46732}{5247}k_3 - \frac{49}{176}k_4 - \frac{5103}{18656}k_5\right) \tag{13}$$

$$k_7 = hf\left(x_i + h, y_i + \frac{35}{384}k_1 + \frac{500}{1113}k_3 + \frac{125}{192}k_4 - \frac{2187}{6784}k_5 + \frac{11}{84}k_6\right) \tag{14}$$

5th-Order Solution [10].

$$k_7 = hf\left(x_i + h, y_i + \frac{35}{384}k_1 + \frac{500}{1113}k_3 + \frac{125}{192}k_4 - \frac{2187}{6784}k_5 + \frac{11}{84}k_6\right) \tag{15}$$

3. Result and Discussion

In this chapter, the results of numerical simulations using the Law of Cooling presented by Newton and the modified version are presented. The paper applies two numerical techniques: RK4 and Dormand-Prince, to simulate the cooling of an object. The numerical results obtained are then compared with the experimental results to analyse the excellence of the techniques. The comparison data is based on an experiment where a stew cheesecake is cooled; the temperature of the object is measured at different time intervals between 0 and 120 minutes. Table 1 describes the experimental result and the exact solution found in the study conducted by [6].

Table 1 Experimental and Exact solution results [6]

Time (minutes)	Experimental Temperature (℃)	Exact Temperature (℃)
0	98.9	98.9
5	75.9	86.545
15	57	68
30	45.1	50.76
45	39.5	41.141
60	35.3	35.774
75	32	32.779
90	31.3	31.109
105	30.6	30.177
120	29.7	29.656

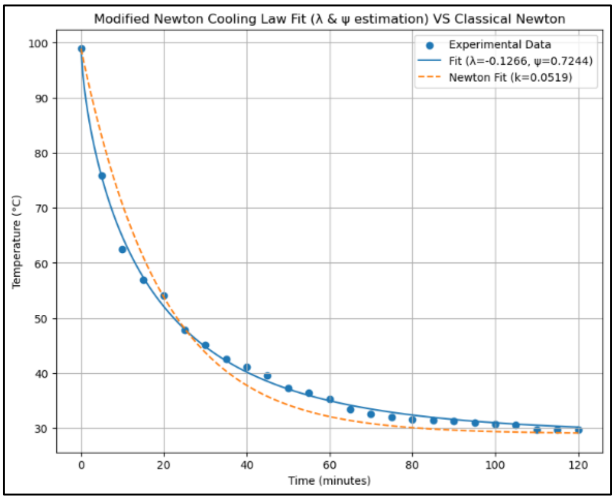
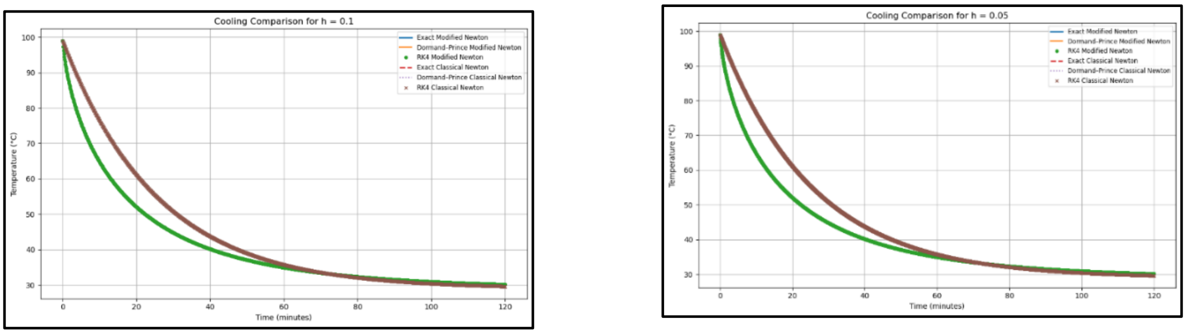


Fig. 1 Comparison of Experimental Data, Fitted Modified Newton's Law of Cooling Model and Fitted classical Newton's Law of Cooling Model

The evaluation commenced with the first step, which involved fitting the modified Newtons Law of Cooling model to the experimental data in order to estimate the parameters of the model $\boldsymbol{\lambda}$ and ψ. Experimental time and temperature data were plotted to fit the curve, and the estimated parameters were then applied to create the fitted model. Based on the results of the curve fitting as shown in Fig. 1, the modified Newtons Law of Cooling model fit the experimental temperature data very well in the cooling period. The fitted temperatures were always near the measured temperatures, which shows that the estimated parameters effectively describe the cooling behaviour of the object. This indicates that the modified model is more appropriate to represent the experimental cooling process than the classical Newton's Law of Cooling.



(a) (b)

Fig. 2 Results for comparison at (a) h=0.1; (b) h=0.05

Fig. 2 shows the temperature profiles obtained using the exact solution, RK4 and Dormand–Prince methods for both the modified and classical Newton's Law of Cooling models with step sizes h=0.1 and h=0.05, respectively. The figures presented the temperature variation from 0 to 120 minutes, starting from the initial temperature and decreasing over time. All results are plotted on the same graph to allow comparison of the temperature values produced by each method at corresponding time points.

Table 2 Table of results for comparison h=0.1

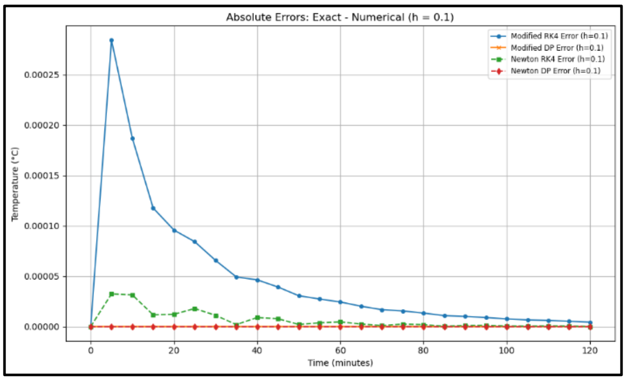
Time (minutes)	Exact Solution (Modified)	RK4 (Modified)	Dormand-Prince (Modified)	Exact Solution (Classical)	RK4 (Classical)	Dormand-Prince (Classical)
0	98.9000	98.9000	98.9000	98.9000	98.9000	98.9000
5	75.5665	75.5663	75.5665	86.5449	86.5449	86.5449
15	57.4121	57.4121	57.4121	68.0002	68.0002	68.0002
30	44.7942	44.7941	44.7942	50.7598	50.7598	50.7598
45	38.5049	38.5048	38.5049	41.1407	41.1407	41.1407
60	34.9864	34.9863	34.9864	35.7738	35.7738	35.7738
75	32.8896	32.8896	32.8896	32.7794	32.7794	32.7794
90	31.5868	31.5868	31.5868	31.1087	31.1087	31.1087
105	30.7523	30.7523	30.7523	30.1765	30.1765	30.1765
120	30.2051	30.2051	30.2051	29.6564	29.6564	29.6564

Table 3 Table of results for comparison h=0.05

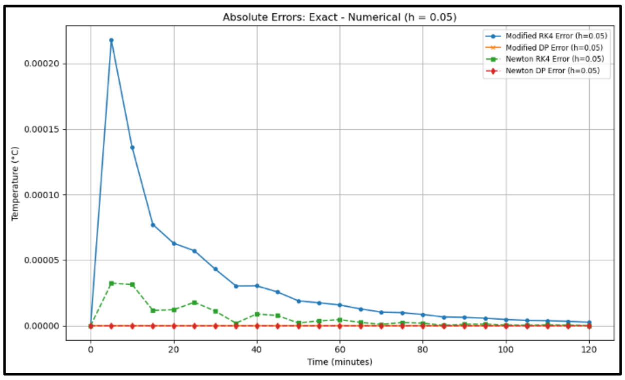
Time (minutes)	Exact Solution (Modified)	RK4 (Modified)	Dormand-Prince (Modified)	Exact Solution (Newton)	RK4 (Newton)	Dormand-Prince (Newton)
0	98.9000	98.9000	98.9000	98.9000	98.9000	98.9000
5	75.5665	75.5662	75.5665	86.5449	86.5449	86.5449
15	57.4121	57.4120	57.4121	68.0002	68.0002	68.0002
30	44.7942	44.7941	44.7942	50.7598	50.7598	50.7598
45	38.5049	38.5048	38.5049	41.1407	41.1407	41.1407
60	34.9864	34.9863	34.9864	35.7738	35.7738	35.7738
75	32.8896	32.8896	32.8896	32.7794	32.7794	32.7794
90	31.5868	31.5868	31.5868	31.1087	31.1087	31.1087
105	30.7523	30.7523	30.7523	30.1765	30.1765	30.1765

120	30.2051	30.2051	30.2051	29.6564	29.6564	29.6564

Based on the results presented in Tables 2 and Table 3 and illustrated in Fig. 2, both RK4 and Dormand–Prince methods successfully approximated the temperature profiles for the classical and modified Newton's Law of Cooling models. The numerical solutions closely followed the exact solutions throughout the cooling duration from 0 to 120 minutes for both step sizes h=0.1 and h=0.05. This implied that both the numerical techniques could effectively solve the governing differential equations of the cooling process. The detail explanation will be explained in error analysis part below.



(a)



(b)

Fig. 3 Error comparison of RK4 and Dormand-Prince with exact solution for (a) h=0.1; (b) h=0.05

Fig. 3 shows the absolute error of the RK4 and Dormand–Prince methods with step sizes h=0.1 and h=0.05, respectively. The error values are obtained by subtracting the exact solution from the numerical solutions produced by each method, followed by taking the absolute value. The figures displayed the absolute error variation with respect to time for both the modified and classical Newton's Law of Cooling models.

Table 4 Error comparison of RK4 and Dormand-Prince with exact solution for h=0.1

Time (minutes)	RK4 (Modified)	Dormand-Prince (Modified)	RK4 (Newton)	Dormand-Prince (Newton)
0	0×10^{0}	0×10^{0}	0×10^{0}	0×10^{0}
5	2.841240×10^{-4}	3.657462×10^{-9}	3.240383×10^{-5}	1.421085×10^{-14}
15	1.176406×10^{-4}	2.230060×10^{-9}	1.163704×10^{-5}	0×10^{0}
30	6.566320×10^{-5}	1.239741×10^{-9}	1.112833×10^{-5}	4.973799×10^{-14}
45	3.924271×10^{-5}	7.461196×10^{-10}	7.759690×10^{-6}	3.552714×10^{-14}
60	2.435130×10^{-5}	4.699103×10^{-10}	4.617184×10^{-6}	2.842171×10^{-14}
75	1.548766×10^{-5}	3.053202×10^{-10}	2.414639×10^{-6}	4.263256×10^{-14}
90	1.002397×10^{-5}	2.030447×10^{-10}	1.077569×10^{-6}	7.105427×10^{-15}
105	6.571842×10^{-6}	1.375575×10^{-10}	3.506394×10^{-7}	1.421085×10^{-14}
120	4.350161×10^{-6}	9.460166×10^{-11}	5.808687×10^{-12}	1.065814×10^{-14}

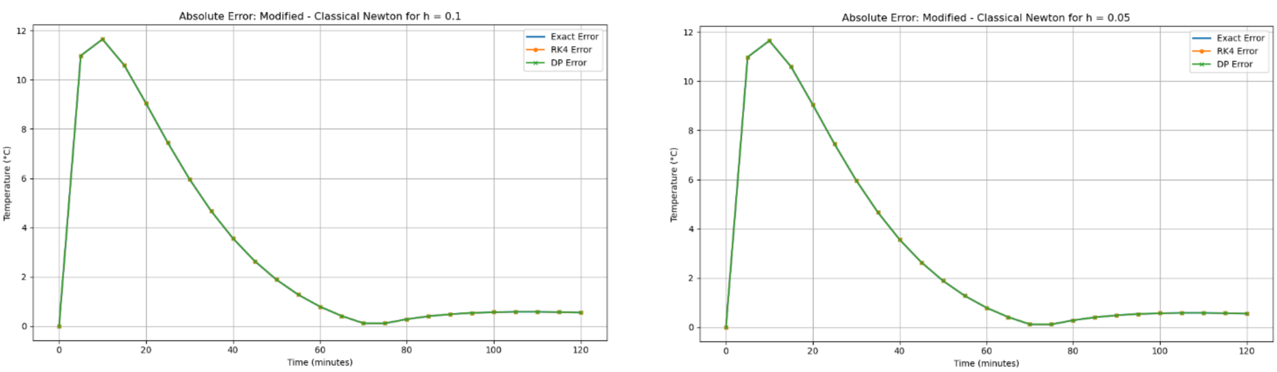
Table 5 Error comparison of RK4 and Dormand-Prince with exact solution for h=0.05

Time (minutes)	RK4 (Modified)	Dormand-Prince (Modified)	RK4 (Newton)	Dormand-Prince (Newton)
0	0×10^{0}	0×10^{0}	0×10^{0}	0×10^{0}
5	2.176949×10^{-4}	3.679759×10^{-9}	3.240385×10^{-5}	5.684342×10^{-14}
15	7.710913×10^{-5}	2.245194×10^{-9}	1.163708×10^{-5}	4.263256×10^{-14}

30	4.313199×10^{-5}	1.248125×10^{-9}	1.112838×10^{-5}	4.263256×10^{-14}
45	2.568350×10^{-5}	7.510579×10^{-10}	7.759728×10^{-6}	7.105427×10^{-15}
60	1.581144×10^{-5}	4.730438×10^{-10}	4.617213×10^{-6}	1.421085×10^{-14}
75	9.938951×10^{-6}	3.073595×10^{-10}	2.414659×10^{-6}	6.394885×10^{-14}
90	6.333769×10^{-6}	2.044231×10^{-10}	1.077582×10^{-6}	2.131628×10^{-14}
105	4.072062×10^{-6}	1.384706×10^{-10}	3.506482×10^{-7}	1.065814×10^{-14}
120	2.631033×10^{-6}	9.524115×10^{-11}	2.167155×10^{-13}	1.776357×10^{-14}

The error analysis presented in Table 4 and Table 5 and illustrated in Fig. 3 show that the absolute errors for both RK4 and Dormand–Prince methods were very small across all time intervals. The errors were obtained by subtracting the exact solution from the numerical solutions and taking the absolute value. In both step sizes, the Dormand-Prince method yielded effective error values that were zero at most of the time whereas the RK4 method yielded small error values that were not zero. This demonstrated that the Dormand–Prince method yielded higher numerical accuracy compared to the RK4 method for both the classical and modified models.

A comparison between the two step sizes showed that reducing the step size from h=0.1 to h=0.05 resulted in slightly smaller errors for the RK4 method, as shown in Table 4 and Table 5. The Dormand–Prince method, however, exhibited the same behaviour with almost zero values of the errors in both step sizes, showing the accuracy of this method to be less sensitive to the choice of step size in the experiment.



(a) *(b)*

Fig. 4 Absolute error comparison between modified and classical Newton's Law of Cooling for (a) $\boldsymbol{h}$=0.1; (b) h=0.05

Fig. 4 shows the temperature profiles obtained using the exact solution, RK4 and Dormand–Prince methods for both the modified and classical Newton's Law of Cooling models with step sizes h=0.1 and h=0.05, respectively.

Table 6 Absolute error comparison between modified and classical Newton's Law of Cooling

Time (minutes)	Exact Solution ($h = 0.1$)	RK4 ($h = 0.1$)	Dormand-Prince ($h = 0.1$)	Exact Solution ($h = 0.05$)	RK4 ($h = 0.05$)	Dormand-Prince ($h = 0.05$)
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
5	10.978453	10.978705	10.978453	10.978453	10.978638	10.978453
15	10.588022	10.588128	10.588022	10.588022	10.588087	10.588022
30	5.965686	5.965740	5.965686	5.965686	5.965718	5.965686
45	2.635855	2.635887	2.635855	2.635855	2.635873	2.635855
60	0.787463	0.787483	0.787463	0.787463	0.787474	0.787463
75	0.110190	0.110177	0.110190	0.110190	0.110182	0.110190
90	0.478109	0.478100	0.478109	0.478109	0.478103	0.478109

105	0.575796	0.575790	0.575796	0.575796	0.575793	0.575796
120	0.548659	0.548655	0.548659	0.548659	0.548656	0.548659

The differences between the modified and classical Newton's Law of Cooling models were analysed using the results presented in Table 6 and illustrated in Fig. 4. The absolute differences were obtained by subtracting the classical model results from the modified model results for the exact solution, RK4, and Dormand–Prince methods, followed by taking the absolute value. These findings indicated that the difference between the two models was bigger in the initial stages of cooling and faded out with time. This behaviour reflected the difference in cooling behaviour predicted by the two models as the object temperature approached the ambient temperature.

4. Conclusion

The RK4 and the Dormand-Prince method numerically solved the classical and the modified Newton's Law of Cooling. The results showed that both numerical models had the ability to model the cooling behaviour of the object appropriately during the period of time. The comparison of the accuracy demonstrated that in both cases of the step sizes, the Dormand-Prince technique produced smaller values of the absolute error than the RK4 technique. The step size analysis of the given research indicates that the step size is significant to the accuracy of the numerical mechanism employed to model the cooling process and that, the RK4 method of modelling the cooling process is highly accurate with smaller step sizes, though the Dormand-Prince method is highly accurate irrespective of the size of the step. The comparison between the classical and modified model of the Newton Law of Cooling showed that the modified model provided significantly different results in predicting temperature in comparison to the classical model, particularly the first stages of cooling. These results verified that the modified Newton's Law of Cooling model and the numerical methods used worked well in the modelling of temperature decay.

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Conflict of Interest

Authors declare that there is no conflict of interests regarding the publication of the paper.

Author Contribution

The authors confirm contribution to the paper as follows: ***study conception and design:*** *Nurul Nazirah Zulkifli, Norzuria Ibrahim;* ***data collection:*** *Nurul Nazirah Zulkifli;* ***analysis and interpretation of results:*** *Nurul Nazirah Zulkifli, Norzuria Ibrahim;* ***draft manuscript preparation:*** *Nurul Nazirah Zulkifli, Norzuria Ibrahim. All authors reviewed the results and approved the final version of the manuscript.*

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