ENHANCED KNOWLEDGE IN SCIENCES AND TECHNOLOGY
e-ISSN: 2773-6385

EKST

Vol. 6 No. 1 (2026) 75-81
https://publisher.uthm.edu.my/periodicals/index.php/ekst

Solving Elliptic Boundary Value Problems using Finite Difference Method in Steady-State Heat Conduction

Nurul Amira Adzazzuddin[1], Fazlina Aman[1*]

[1] *Department of Mathematics and Statistics, Faculty of Applied Sciences and Technology, UTHM Kampus Cawangan Pagoh, Hab Pendidikan Tinggi Pagoh, KM 1, Jalan Panchor, 84600 Pagoh, Muar, Johor, MALAYSIA.*

*Corresponding Author: fazlina@uthm.edu.my
DOI: https://doi.org/10.30880/ekst.2026.06.01.010

Article Info

Received: 2 January 2026
Accepted: 5 June 2026
Available online: 9 July 2026

Keywords

Finite Difference Method, Elliptic Boundary Value Problem, Steady-State Heat Conduction, Numerical Solution, Poisson Equation, MATLAB Implementation, Convergence Analysis

Abstract

This paper explores the use of finite difference method (FDM) for the solution of an elliptic boundary value problem arising in one-dimensional steady-state heat conduction. The main goals are developing the governing equation into a computational model and test the accuracy of the model using a computational approach. The suggested strategy will address the problem of forecasting temperature distributions when no analytical solution exists. A second-order central difference scheme is used to discretize the steady-state heat conduction equation, and the problem is reduced to a series of linear algebraic equations, which provide good approximations and errors of less than 3% even on coarse grids, and approach machine accuracy with finer grids. Convergence analysis shows L_2 and L_∞ norms to test the second-order accuracy of the method. The research findings indicate that FDM is a simple, consistent, and efficient numerical method for solving steady-state heat conduction problems, providing a robust basis for thermal analysis.

1. Introduction

Boundary value problems (BVPs) are a key scientific and engineering problem because many physical systems involve partial differential equations with given boundary conditions. The equations of steady-state heat conduction are elliptic partial differential equations (typically Laplace or Poisson equations), and the solution is not time-dependent [1] and [2]. These equations play a significant role in the temperature setting of thermal equilibrium in electronic cooling, building insulation, and material processing [3] and [4]. Such problems are commonly modelled mathematically using the Poisson equation, which describes internal heat generation in homogeneous media [5].

Analytical methods of solving elliptic BVPs typically apply only to those problems whose geometry and idealized boundary conditions [6]. In real-life engineering situations, however, complex geometries, mixed boundary conditions, or spatially varying material properties are standard, and analytical methods are therefore not feasible [7] and [8]. This has prompted the invention and use of numerical techniques that provide approximate solutions to otherwise unsolvable problems [9]. Among them, the finite difference (FDM) method is simple, easy to implement, and computationally efficient, particularly in structured domains [10] and [11].

In the finite difference method, the differential domain is discretized into a grid of points, and derivatives are approximated using difference equations; the resulting system of difference equations is derived from the governing partial differential equation of state, yielding a system of algebraic equations [12] and [13]. This method is efficient in one-dimensional steady-state heat conduction equations, to which it has been extensively verified to be correct [14] and [15]. The latest works have generalized FDM to the schemes of higher order and adaptive grids to enhance the accuracy and efficiency in the complicated configurations [16] and [17]. Moreover, the use of

FDM in combination with software such as MATLAB has increased its ease of use and accessibility in both academia and the workplace [18] and [19]. FDM remains a pillar of elliptic numerical analysis despite more sophisticated methods such as physics-informed neural networks, as it has demonstrated durability and well-understood convergence behaviour [20] and [21].

The main issue this paper addresses is the accurate forecasting of temperature fields in steady-state heat-conduction problems where analytical solutions are not readily available. Although FDM is established, its practical implementation, validation, and convergence analysis to particular Poisson-type problems are subject to thorough investigation, especially in the case of educational analysis and preliminary analysis of engineering. The proposed study thus addresses a canonical one-dimensional problem where Dirichlet boundary conditions are applied in order to show systematically how the method is applied, how accurate it is, and how it converges.

This study aims at achieving two things: one being the development of a finite difference model of a one-dimensional elliptic boundary value problem due to steady-state heat conduction, and the second is to verify the accuracy of the model by comparing it to an analytical solution, both of which are to be implemented and analysed with MATLAB. The paper also explores the properties of convergence of the method using L_2 and L_∞ error norms, and hence, gives a straightforward guideline to the numerical solution of steady-state thermal problems.

2. Methodology

2.1 Governing Equations of the Elliptic Boundary Value Problem

Steady-state heat conduction in a homogeneous medium with internal heat production can be described by the Poisson equation [22]:

$$\frac{d^2T}{dx^2} = -f(x), \qquad 0<x<1, \tag{1}$$

$T(x)$ is a temperature distribution, and $f(x)$ is the internal term of the heat source. The spatial domain $0<x<1$ represents a simplified one-dimensional model of a physical system in which temperature change occurs primarily in one direction.

2.2 Finite Difference Method

Dirichlet conditions are applied at both ends of the domain:

$$T(0)=0, \quad T(1)=0, \tag{2}$$

Modelling discrete temperatures at the boundaries is frequently employed in steady-state heat conduction analysis and provide a physical explanation of thermal constraints [23].

An analytical solution has been provided to facilitate numerical verification, as outlined below.

$$T(x)=e^{-x}x(1-x)\,. \tag{3}$$

This function satisfies the established boundary conditions and exhibits a continuous temperature profile throughout the domain. The analytical solution is differentiated twice with respect to x producing the corresponding source term $f(x)$ and the governing equation is meticulously obeyed. The method of verification is standard to numerical analysis of elliptic problems of the second kind [24] and [25].

The finite difference method is used through the discretization of the spatial domain into a system of uniformly spaced grid points. The N equal subintervals of $[0,1]$ form a grid with spacing $h=1/N$. Each grid point is approximated by a discrete nodal value that represents the temperature at that point.

The central difference scheme is used to approximate the second derivative of temperature with respect to space,

$$\left.\frac{d^2T}{dx^2}\right|_{x=x_i} \approx \frac{T_{i+1}-2T_i+T_{i-1}}{h^2}, \tag{4}$$

which is second-order accurate and used in the steady-state heat conduction and Poisson-type problems [26] and [27]. Replacing this approximation in the governing equation yields a system of linear algebraic equations among the nodal temperatures.

The boundary conditions are addressed by fixing the temperature values of the boundary nodes. The resulting system is of a tridiagonal coefficient matrix, thereby allowing for an efficient numerical solution. MATLAB is chosen as the computing system because it offers powerful features for operating on matrices and stable linear solvers [28].

Both coarse and refined grids are numerically simulated to investigate the effect of grid resolution on the accuracy of the solution and convergence behaviour.

3. Results and Discussion

The comparison of numerical values of temperature at the grid points in the domain with the analytical value is reflected in Table 1. Both the absolute and percentage errors are provided to measure the accuracy of the numerical estimation.

Overall, the numerical findings are in good agreement with the analytical solution. All interior nodes have a percentage error of less than 3%, which means that the finite difference method is a relatively accurate approximation, even when a coarse grid discretization is used. It shows the potential of the method to model the overall behaviour and trend of the temperature field in steady-state problems of heat conduction.

It can be noted that both the absolute and the percentage errors steadily grow with the distance of the spatial coordinate to the right boundary of the domain. This behaviour is often linked to discretization error in finite difference methods. This effect is caused by the Dirichlet boundary conditions, which are enforced. It is also common to find that numerical error accumulation along the boundary is high on coarse grids, and it is not an indication of numerical method instability.

Although there is a discretisation error, the numerical solution has been able to reproduce the shape and magnitude of the analytical temperature profile throughout the domain. The error values are relatively small, indicating that the second-order central difference approximation used in this research is consistent and reliable. These findings indicate that additional grid refinement would produce an even more minor difference between numerical and analytical solutions, as predicted by the convergence properties of the finite difference method theory.

Overall, the comparison in Table 1 confirms the validity of the finite difference method for solving the one-dimensional elliptic boundary value problem examined in this paper and provides a sound foundation for further analysis of convergence.

To further investigate the numerical behaviour of the finite difference method, the calculations were also performed with various grid step sizes. The numerical solution with decreasing grid spacing shows minor deviations from the analytical solution. This means that the higher the step size, the less accurate the approximation of the desired numerical value, a key property of a consistent finite difference scheme. The fact that several different step sizes were used to evaluate the numerical method also makes the evaluation robust. It shows that it can indeed be used to solve steady-state heat conduction problems. In this study, numerical simulations were carried out using several grid step sizes, namely $h = 0.2, 0.1, 0.05$, and 0.025, to examine the effect of grid refinement on the numerical solution.

Table 1 Comparison between numerical and analytical temperature values at selected nodes

Node ($\boldsymbol{i}$)	Step size ($\boldsymbol{h}$)	Position ($\boldsymbol{x_i}$)	Numerical ($\boldsymbol{T_i}$)	Analytical $\boldsymbol{T(x_i)}$	Absolute Error	Percentage Error (%)
1	0.2	0.2	0.128638	0.130997	0.002359	1.80
2	0.2	0.4	0.157720	0.160877	0.003157	1.96
3	0.2	0.6	0.128886	0.131714	0.021471	2.15
4	0.2	0.8	0.070195	0.071893	0.001698	2.36

Table 2 Finite difference discretization at a single interior node ($\boldsymbol{x} = \mathbf{0.2}$) for different step sizes

Step size ($\boldsymbol{h}$)	**Node ($\boldsymbol{i}$)**	$\boldsymbol{h^2}$	$\boldsymbol{h^2 f(0.2)}$	**Finite difference equation**
0.20	1	0.040000	-0.02738	$\boldsymbol{-2T_1 + T_2 + T_0 = -0.02738}$
0.10	2	0.010000	-0.00685	$\boldsymbol{-2T_2 + T_3 + T_1 = -0.00685}$
0.05	4	0.002500	-0.00171	$\boldsymbol{-2T_4 + T_5 + T_3 = -0.00171}$
0.025	8	0.000625	-0.00043	$\boldsymbol{-2T_8 + T_9 + T_7 = -0.00043}$

Table 2 shows how the grid step size influences the behaviour of the discretisation of the finite difference method with one interior node. The larger the step size, the larger the magnitude of the discretisation term, and this is a representation of the second-order nature of the central difference approximation. This observation supports the conclusion that grid refinement increases numerical accuracy because smaller step sizes reduce truncation error in the finite-difference formulation. The smooth decrease in the discretisation contribution due to the reduction in the grid spacing provides definite numerical support for the convergence and consistency of the finite difference scheme.

The temperature distribution is calculated using the analytical solution and is compared with the finite difference solution, as shown in Fig. 1. The analytical profile is in excellent agreement with the numerical results across the entire domain. In cases of coarse grid discretization, minor discrepancies are found at the interior nodes, and this is explained by the discretization error. The numerical solution, however, can preserve the magnitude and form of the temperature distribution, thereby proving the accuracy of the finite difference formulation.

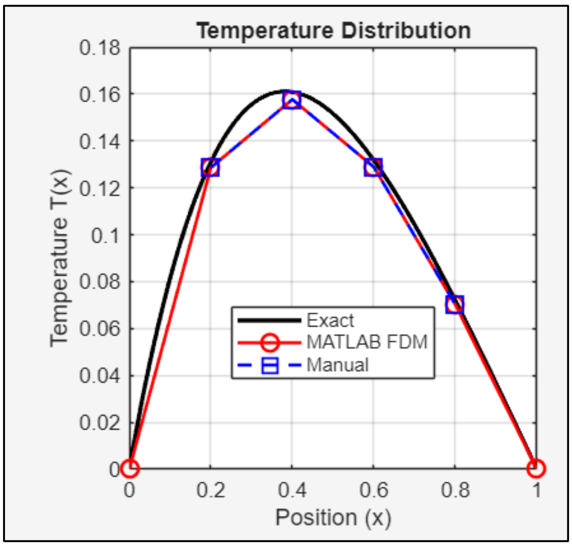

Fig. 1 Temperature distribution

Fig. 2 shows the convergence behaviour of the numerical method using the L_∞ error norm. The error is proportional to the square of the grid spacing, indicating second-order convergence. This behaviour is in line with the theoretical validity of the central difference approximation in the discretization of the governing equation. The convergence findings also confirm the accuracy and consistency of the finite difference approach in the solution of elliptic equations of the steady-state heat conduction of a boundary value problem. The systematic reduction in numerical error with decreasing grid spacing further confirms that grid refinement significantly improves solution accuracy. This behaviour provides clear numerical evidence that the finite difference method converges towards the analytical solution as the mesh is refined.

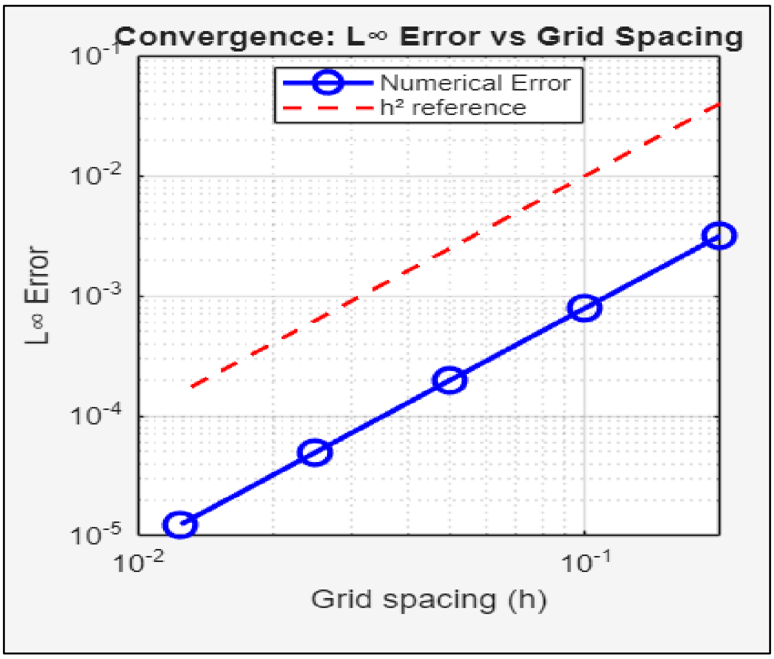

Fig. 2 Convergence plot L_{∞}

4. Conclusion

The paper has managed to use the finite difference method to obtain a solution to a one-dimensional elliptic boundary value problem that comes up in the circumstances of steady-state heat conduction. This was discretized in MATLAB as the Poisson equation was solved with a second-order central difference scheme. The numerical solution has been validated against an analytical solution to assess accuracy and convergence behaviour. This comparison of the numerical temperature with the analytical temperature values showed good agreement throughout the computational domain, with percentage errors of less than 3% encountered when a coarse grid discretization was used. The obtained results can be used to verify that the temperature distribution can be well approximated even with a small number of grid points using the finite difference method. The observed differences were primarily due to discretization error and decreased with grid refinement. The convergence analysis also confirmed the number formulation. The L_{∞} error norm showed the second-order convergence rate of the grid spacing, with the theoretical accuracy of the central difference approximation. This behaviour validates the stability, consistency, and reliability of the finite difference scheme used in this research. Overall, the results indicate that the finite difference method is a straightforward, effective, and accurate numerical technique for solving elliptic boundary value problems in heat conduction at steady-state. The current study is limited to a one-dimensional problem with Dirichlet boundary conditions. However, with further investigation and research, it can be easily generalized to higher-dimensional domains and non-uniform grids, as well as more complex boundary conditions.

Acknowledgement

The authors would like to thank the Faculty of Applied Sciences and Technology, Universiti Tun Hussein Onn Malaysia for its support.

Conflict of Interest

Authors declare that there is no conflict of interest regarding the publication of the paper.

Author Contribution

The authors confirm contribution to the paper as follows: ***study conception and design:*** *Nurul Amira Adzazzuddin;* ***data collection:*** *Nurul Amira Adzazzuddin;* ***analysis and interpretation of results:*** *Nurul Amira Adzazzuddin, Fazlina Aman;* ***draft manuscript preparation:*** *Nurul Amira Adzazzuddin, Fazlina Aman. All authors reviewed the results and approved the final version of the manuscript.*

References

[1] J. Bollati, C. M. Gariboldi, and D. A. Tarzia, “Explicit solutions for distributed, boundary and distributed-boundary elliptic optimal control problems,” *Journal of Applied Mathematics and Computing*, vol. 63, no. 1–2, pp. 1–75, May 2020, doi: 10.1007/s12190-020-01355-2.

[2] R. Z. Zhang and L.-T. Cheng, “A Compact Coupling Interface Method with Second-Order Gradient Approximation for Elliptic Interface Problems,” *Journal of Scientific Computing*, vol. 100, no. 2, Jun. 2024, doi: 10.1007/s10915-024-02587-1.

[3] E. G. Varuvel, A. Sonthalia, F. Aloui, and C. G. Saravanan, “Basics of heat transfer: Conduction,” in *Elsevier eBooks*, 2023, pp. 1–33. doi: 10.1016/b978-0-443-19017-9.00003-9.

[4] P. Das, Md. A. Islam, D. Mondal, and Md. S. Nazim, “Analytical and numerical (FEA) solution for steady-state heat transfer in generic FGM cylinder coated with two layers of isotropic material under convective-radiative boundary conditions,” *Heliyon*, vol. 9, no. 11, p. e21725, Oct. 2023, doi: 10.1016/j.heliyon.2023.e21725.

[5] Md. S. Hossan, T. Datta, and Md. S. Islam, “Combining approach of collocation and finite difference methods for fractional parabolic PDEs,” *Partial Differential Equations in Applied Mathematics*, vol. 12, p. 100921, Sep. 2024, doi: 10.1016/j.padiff.2024.100921.

[6] H. Gao, L. Sun, and J.-X. Wang, “PhyGeoNet: Physics-Informed Geometry-Adaptive Convolutional Neural Networks for solving Parameterized Steady-State PDEs on irregular domain,” *arXiv (Cornell University)*, Apr. 2020, doi: 10.48550/arxiv.2004.13145.

[7] J. A. Rodrigues, “Solving Steady-State elliptic problems in irregular domains using Physics-Informed neural networks and fictitious domain methods,” *Research in Statistics*, vol. 3, no. 1, Aug. 2025, doi: 10.1080/27684520.2025.2542577.

[8] T. G. Grossmann, U. J. Komorowska, J. Latz, and C.-B. Schönlieb, “Can physics-informed neural networks beat the finite element method?,” *IMA Journal of Applied Mathematics*, vol. 89, no. 1, pp. 143–174, Jan. 2024, doi: 10.1093/imamat/hxae011.

[9] A. L. M. A. Magalhães *et al.*, “Numerical resolution of differential equations using the finite difference method in the real and complex domain,” *Mathematics*, vol. 12, no. 12, p. 1870, Jun. 2024, doi: 10.3390/math12121870.

[10] K. Pan, K. Fu, J. Li, H. Hu, and Z. Li, “New Sixth-Order Compact Schemes for Poisson/Helmholtz equations,” *Numerical Mathematics Theory Methods and Applications*, vol. 16, no. 2, pp. 393–409, Apr. 2023, doi: 10.4208/nmtma.oa-2022-0073.

[11] Md. S. H. Mojumder, Md. N. Haque, and Md. J. Alam, “Efficient finite difference methods for the numerical analysis of One-Dimensional heat equation,” *Journal of Applied Mathematics and Physics*, vol. 11, no. 10, pp. 3099–3123, Jan. 2023, doi: 10.4236/jamp.2023.1110204.

[12] J. Hu and L. Ma, “A direct finite element method for elliptic interface problems,” *arXiv (Cornell University)*, Jan. 2024, doi: 10.48550/arxiv.2401.16967.

[13] N. Sahoo, R. Singh, A. Kanaujiya, and C. Cattani, “A new fourth-order compact finite difference method for solving Lane-Emden-Fowler type singular boundary value problems,” *Journal of Computational Science*, vol. 83, p. 102474, Nov. 2024, doi: 10.1016/j.jocs.2024.102474.

[14] O. C. N, “Analytical Solution of Steady State Heat Conduction in a Rectangular Plate and Comparison with the Numerical Finite Difference Method,” *International Journal of Mathematical Research*, vol. 11, no. 1, pp. 10–15, Jun. 2022, doi: 10.18488/24.v11i1.3035.

[15] V. G. De Brito Dos Santos and P. T. G. D. Anjos, “Finite difference method applied in Two-Dimensional heat conduction problem in the permanent regime in rectangular coordinates,” *Advances in Pure Mathematics*, vol. 12, no. 09, pp. 505–518, Jan. 2022, doi: 10.4236/apm.2022.129038.

[16] Y. Ren and S. Zhao, “A FFT accelerated fourth order finite difference method for solving three-dimensional elliptic interface problems,” *Journal of Computational Physics*, vol. 477, p. 111924, Jan. 2023, doi: 10.1016/j.jcp.2023.111924.

[17] S. Milewski, “Direct and Inverse Steady-State Heat Conduction in Materials with Discontinuous Thermal Conductivity: Hybrid Difference/Meshless Monte Carlo Approaches,” *Materials*, vol. 18, no. 18, p. 4358, Sep. 2025, doi: 10.3390/ma18184358.

[18] R. G. Albarak, “Study and comparison of finite difference methods for the numerical solution of partial differential equations,” *Nanotechnology Perceptions*, pp. 208–227, Dec. 2024, doi: 10.62441/nano-ntp.vi.3459.

[19] L. Breedan, *MATLAB implementation of Finite Differences Method for Deep Beams*, Technical University of Cluj-Napoca, 2015. [Online]. Available: https://doi.org/10.13140/RG.2.1.1296.1369

[20] Z. Yang, X. Liu, Z. Zhang, S. Li, and Q. Fang, “Analysis of preheating temperature field characteristics in selective laser sintering,” *Advances in Mechanical Engineering*, vol. 14, no. 1, pp. 1–14, 2022, doi: 10.1177/16878140211072397.

[21] C. Yuan, C. Zhang, and Y. Zhang, "Direct solution of inverse steady-state heat transfer problems by improved coupled radial basis function collocation method," _Mathematics_, vol. 13, no. 9, p. 1423, Apr. 2025, doi: 10.3390/math13091423.