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Shooting Method to Solve Nonlinear Boundary Value Problem in Heat Transfer

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Abstract

This study investigates the application of the shooting method to solve nonlinear boundary value problems (BVP) in heat transfer focusing on a one-dimensional fin with temperature dependent conductivity and conductive-convective-radiative boundary conditions. The objective of this study is to solve the nonlinear BVP using the shooting method, validate the results by comparing with existing literature, and analyses the method's accuracy and computational efficiency. The governing equation is converted into a dimensionless form and transformed into a system of first-order ordinary differential equations. The shooting method, implemented in MATLAB with a fourth-order Runge-Kutta (RK4) method, then iteratively adjusts the unknown initial condition at the fin tip until the boundary condition at the base is satisfied. Numerical solutions are obtained for various values of the convection parameter, $\mathrm{N_c}$, radiation parameter, $\mathrm{N_r}$, and thermal conductivity parameter, λ. The results show that higher radiation led to more heat loss and lower fin temperatures, while increased thermal conductivity helped retain heat. The temperature distributions closely match those reported in previous studies confirming the accuracy of the shooting method. Furthermore, the method demonstrated computational efficiency, converging within 1-6 iterations and requiring less than 0.15 seconds of Central Processing Unit (CPU) time for all cases. All computations were carried out using MATLAB on a computer equipped with an AMD Ryzen 3 3250U with Radeon Graphics processor ,8GB RAM and Windows operating system. In conclusion, the shooting method is an accurate, and efficient numerical tool for solving nonlinear BVPs in heat transfer involving conductive, convective, and radiative effects.

1. Introduction

Nonlinear boundary value problems (BVP) are common in heat transfer modelling, especially when systems involve temperature-dependent thermal properties and combined conductive–convective–radiative boundary conditions. The nonlinearity of these systems arises from the temperature dependence of thermal conductivity in the governing equation while conductive, convective and radiative heat exchange at the boundaries introduce additional nonlinearities. Due to these nonlinearities, reliable and efficient numerical methods in order to achieve accurate solutions are essential.

Numerous studies have been conducted involving nonlinear BVP in heat transfer using difference type of numerical method. For instance, [1] emphasized nonlinear behaviour plays a critical role in applications such as geothermal energy systems, where both material properties and boundary conditions are strongly temperature-dependent. These interdependencies introduce nonlinearities not only into the governing partial differential equations but also into the boundary constraints, necessitating accurate and stable numerical techniques for effective thermal analysis and system design. Also, [2] studied heat conduction with temperature-dependent thermal conductivity and boundary conditions, finding that only strong numerical techniques yield accurate results. Similarly, [3] and [4] show how numerical tools are essential to solve nonlinear equation that arise due to radiation effects and constant and temperature-dependent conductivity. Their studies highlight that numerical methods are essential when analysing heat transfer problems that involve nonlinearities in both the governing equation and the boundary conditions.

Numerical methods like differential transformation method (DTM) are used to solve the nonlinear governing equation in [5] studies and its accuracy confirmed by comparing with exact solution. [6] used Least-squares spectral collocation method (LSSCM) to study heat transfer in a moving porous plate exposed to both convective and radiative effects. [7] used Galerkin method to solve radiative-conductive heat transfer problem, in order to address the nonlinearities by radiation effects boundaries. [8] used meshless weighted least squares method with Kirchoff transformation to handle radiation-dominated boundaries.

Among these numerical methods, the shooting method is a popular numerical technique known for its simplicity and adaptability in transforming boundary value problems (BVPs) into more manageable initial value problems (IVPs). Studies by [9] and [10] demonstrated and validate the accuracy of shooting method across a variety of nonlinear problems in heat transfer. In [11] used the Runge-Kutta method with a shooting algorithm to solve nonlinear heat conduction problems in biological systems. Their work, which focused on human-head models with convective boundaries, demonstrated the method's effectiveness for complex, nonlinear fluid and thermal sciences.

Despite this advancement, the shooting method performance in solving nonlinear boundary value problems for one-dimensional fins with simultaneous conductive, convective, and radiative boundary conditions, as well as temperature-dependent conductivity, has not been thoroughly explored. A systematic evaluation of its computational efficiency, convergence behaviour, and accuracy when addressing these combined nonlinear effects is needed. To address this gap, this study applies shooting method, implemented with a fourth-order Runge-Kutta (RK4) scheme in MATLAB to solve the nonlinear BVP governing heat transfer in a conductive–convective–radiative fin. This study aims to solve the nonlinear BVP using the shooting method, validate the results by comparison with existing literature, and analyses the method’s accuracy and computational efficiency.

Nomenclature	
A	Cross-sectional area of the fin
h	Convective heat transfer coefficient
$k(T)$	Temperature-dependent thermal conductivity
L	Axial length
l	Dimensionless fin parameter
N_c	Convection parameter
N_r	Radiation parameter
P	Perimeter of the fin cross-section
T_a	Ambient temperature for convection
T_b	Base temperature
T_s	Sink Temperature
X	Axial distance measured from the fin's tip
x	Dimensionless axial distance measured from the tip
Greek symbol	
σ	Stefan-Boltzmann constant
ε	Emissivity of the surface
λ	Dimensionless thermal conductivity
θ	Dimensionless temperature distribution
θ_0	Dimensionless temperature at the fin tip

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Subscript

a	Ambient
b	Base
c	Convection
r	Radiation
s	sink

2. Research Methodology

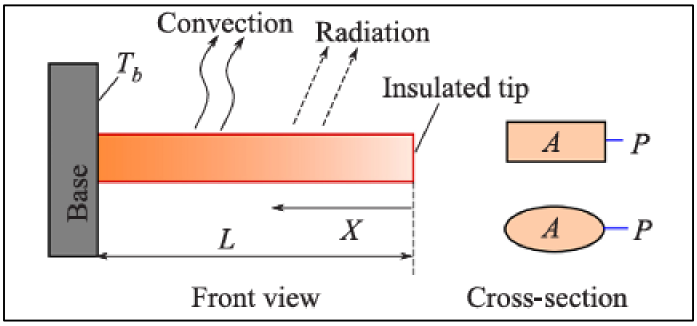

Fig 1 Physical model of a conductive-convective-radiative fin [11]

A rectangular solid fin with temperature-dependent thermal conductivity is considered. The fin loses heat to its surroundings through both convection and radiation, leading to a nonlinear governing equation. The analysis is based on several key assumptions including steady-state, one-dimensional heat conduction, no internal heat sources, a constant ambient temperature, an insulated fin tip, a linear temperature dependence of thermal conductivity, a constant convective heat transfer coefficient, and radiative heat loss that follows the Stefan–Boltzmann law. This physical problem was previously studied by [11] using a semi-analytical approach based on Taylor series expansion, whereas this study will use shooting method.

By referring to [11], the governing equation based on the assumptions are:

$$\frac{d}{dX}\left(k(T)A\frac{dT}{dX}\right)-Ph\left(T-T_a\right)-\varepsilon\sigma P\left(T^4-T_s^4\right) \qquad 0<X<L \tag{1}$$

where, A is a cross-sectional area and X is the location measured from the fin tip. Then, $Ph(T-T_a)$ represent the heat loss due to convection from the fin surface to the surroundings where P is the perimeter of the fin cross-section, h is the convective heat transfer coefficient, T_a is the ambient temperature and T is a temperature of the fin. The term $\varepsilon\sigma(T^4-T_s^4)$ represent the radiative heat loss from the fin surface, which follow the Stefan-Boltzmann law, where ε is the surface emissivity, σ is the Stefan-Boltzmann constant and T_s is sink temperature. These convective and radiative terms introduce nonlinearity into the governing equation. With boundary condition:

$$\frac{dT}{dX}(0)=0,\ \ T(L)=T_b \tag{2}$$

where $\frac{dT}{dX}(0)=0$ represent the fin tip is insulated, so no heat flows out at the tip while $T(L)=T_b$ represent fin base maintained at a constant base temperature. *k(T)* is the temperature-dependent thermal conductivity of the material:

$$k(T)=k\left[1+\lambda'\left(T-T_a\right)\right] \tag{3}$$

where T_a denotes the ambient temperature for convection, k represents the thermal conductivity when the temperature inside the fin equals to the ambient temperature, and λ' is a parameter related to the temperature change. This linear relationship assumes a thermal conductivity that changes proportionally with temperature. This assumption contributes to the nonlinearity of the governing equation.

To generalise the solution, the following non-dimensional variables and parameters are introduced:

$$x=\frac{X}{L},\ \theta=\frac{T-T_a}{T_b-T_a},\ l=\frac{LP}{A},\ N_c=l^2\frac{hA}{kP},\ \ N_r=l^2\frac{\varepsilon\sigma AT_b^3}{kP},\ \ \lambda=\lambda'\left(T_b-T_a\right) \tag{4}$$

The dimensionless axial distance x is representing the position along the fin by its length, L while the dimensionless temperature θ represent the ratio of the local temperature difference to the base-to-ambient temperature difference and l is a dimensionless fin parameter that represent the fin's surface to cross-sectional area ratio. N_c is the dimensionless convection parameter, N_r is the dimensionless radiation parameter and λ is the dimensionless thermal conductivity parameter.

By substituting the nondimensional parameter into equation (1) the governing equations becomes:

$$\frac{d}{dx}\left[(1+\lambda\theta)\frac{d\theta}{dx}\right]-N_c\theta-N_r\theta^4=0,\qquad 0<x<1 \tag{5}$$

with boundary conditions

$$\frac{d\theta}{dx}(0)=0,\ \ \theta(1)=1 \tag{6}$$

The shooting method is used to solve the nonlinear BVP given by (5) to (6). The process involves converting the BVP into an initial value problem (IVP), solving it numerically, and iteratively adjusting the guessed initial condition until the boundary condition at the other end is satisfied.

The second-order ordinary differential equation (ODE) is first rewritten as:

$$\frac{d}{dx}\left[(1+\lambda\theta)\frac{d\theta}{dx}\right]=(1+\lambda\theta)\frac{d^2\theta}{dx^2}+\lambda\left(\frac{d\theta}{dx}\right)^2 \tag{7}$$

then, convert into first ODE by let:

$$y_1=\theta, y_2=\frac{d\theta}{dx} \tag{8}$$

substitute (8) into (7)

$$\left(1+\lambda\, y_1\right)\frac{dy_2}{dx}+\lambda\, y_2^2-N_c\, y_1-N_r\, y_1^4=0 \tag{9}$$

The system becomes:

$$\frac{dy_1}{dx}=y_2 \tag{10}$$

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$$\frac{dy_2}{dx} = \frac{N_c y_1 + N_r y_1^4 - \lambda y_2^2}{1 + \lambda y_1} \tag{11}$$

then the boundary conditions in (6) becomes:

$$y_2(0) = 0, y_1(1) = 0 \tag{12}$$

Using the initial guess based on the [11] and physical reasoning, the system of first ODE is solved from $x = 0$ to $x = 1$ using Runge-Kutta (RK4) method and implemented in MATLAB to obtain the numerical solution for the nonlinear conductive-convective-radiative fin with thermal conductivity.

3. Result and Discussion

Fig. 2 shows the dimensionless temperature distribution $\boldsymbol{\theta(x)}$ along the fin for $\boldsymbol{N_c = 0.5}$ with $\boldsymbol{N_r = 0, 0.5, 1}$ and $\boldsymbol{\lambda = 0, 1, 2, 4}$. The temperatures increase monotonically from the tip $\boldsymbol{x = 0}$ to base $\boldsymbol{x = 1}$, shows that it satisfies the boundary condition $\boldsymbol{\theta(1) = 1}$. As shown in Table 1, higher values of $\boldsymbol{\lambda}$ lead to elevated temperatures, especially near the tip. For example, with $\boldsymbol{N_r = 0}$, the tip temperature $\boldsymbol{\theta(0)}$ rises from 0.793278 at $\boldsymbol{\lambda = 0}$ to 0.951072 at $\boldsymbol{\lambda = 4}$, reflecting enhanced conductive heat transfer that reduces thermal drop. Conversely, increasing the radiation parameter, Nr, results in lower tip temperatures across all $\boldsymbol{\lambda}$ values. This is due to intensified radiative heat loss, especially in regions with limited conductive heat supply from the base. For example, at $\boldsymbol{\lambda = 0}$, $\boldsymbol{\theta(0)}$ decreases from 0.793278 at $\boldsymbol{N_r = 0}$ to 0.675478 at $\boldsymbol{N_r = 1}$. These trends align with the expected physical behavior of conductive-convective-radiative fins and the curve of the graph obtain match closely with the findings in Fig.4, thus validating the accuracy and reliability of the shooting method in solving this nonlinear boundary value problem.

Fig. 3 shows the dimensionless temperature distribution $\boldsymbol{\theta(x)}$ along the fin for $\boldsymbol{N_c = 2.0}$ with $\boldsymbol{N_r = 0, 0.5, 1}$ and $\boldsymbol{\lambda = 0, 1, 2, 4}$. The temperatures increase monotonically from the tip $\boldsymbol{x = 0}$ to base $\boldsymbol{x = 1}$, shows that it satisfies the boundary condition $\boldsymbol{\theta(1) = 1}$. Compared to the case where $\boldsymbol{N_c = 0.5}$, temperatures are noticeably lower across all parameter combinations, reflecting the stronger cooling effect due to convection. However, higher $\boldsymbol{\lambda}$ values still result in higher temperatures along the fin, especially near the tip. This is because increased thermal conductivity enhances heat conduction, even with strong convection. Table 2 further reveals that increasing the radiation parameter, $\boldsymbol{N_r}$, consistently reduces the tip temperature, $\boldsymbol{\theta(0)}$, for all $\boldsymbol{\lambda}$ values. For example, at $\boldsymbol{\lambda = 0}$, $\boldsymbol{\theta(0)}$ decreases from 0.459098 at $\boldsymbol{N_r = 0}$ to 0.435646 at $\boldsymbol{N_r = 1}$. This trend continues for higher $\boldsymbol{\lambda}$ values, confirming that radiative heat loss significantly affects the thermal profile, particularly in regions with limited conductive heat input from the base. These results demonstrate that increased radiation promotes heat loss and reduces temperatures, while enhanced thermal conductivity aids in heat retention. These physical behaviors are accurately captured by the shooting method, and the temperature distributions closely match the graph in Fig.5, validating the method's accuracy and reliability in solving nonlinear boundary value problems in heat transfer.

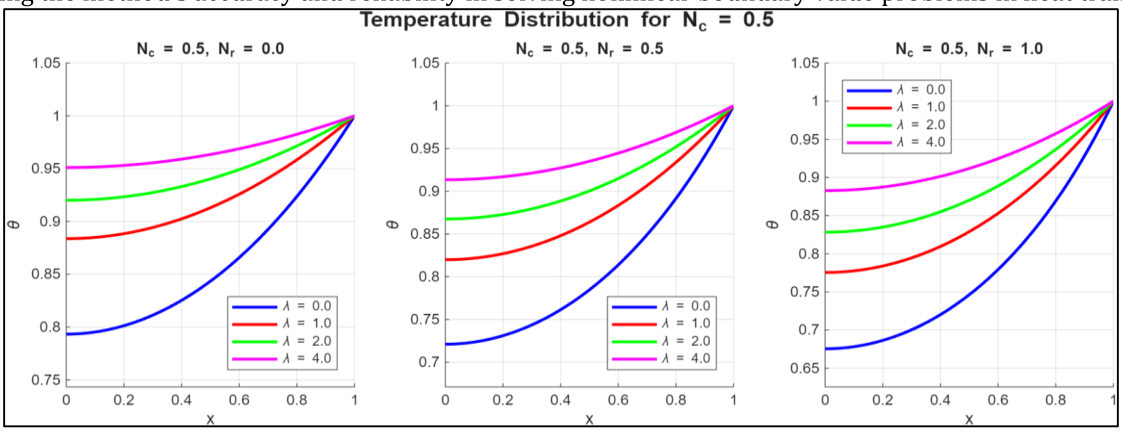

Fig. 2 Temperature distribution for $\boldsymbol{N_c = 0.5}$ and $\boldsymbol{N_r = 0, 0.5, 1}$

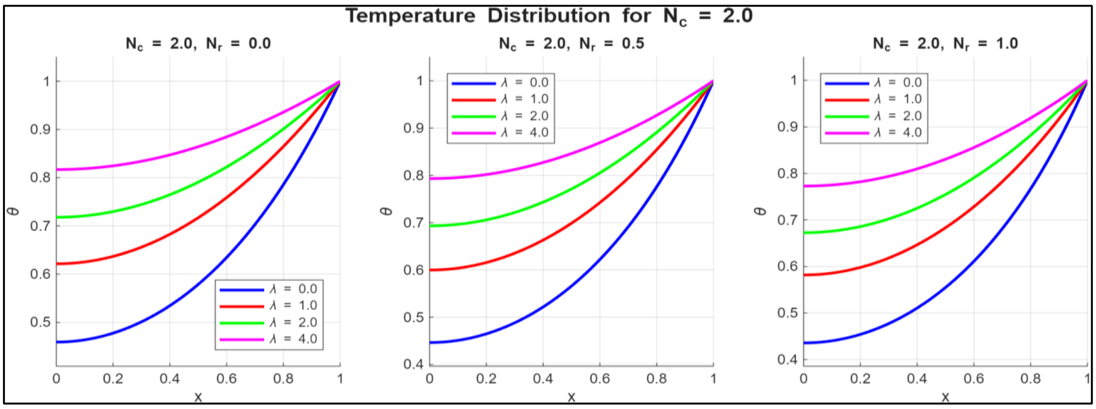

Fig. 3 Temperature distribution for $\boldsymbol{N_c = 2}$ and $\boldsymbol{N_r = 0, 0.5, 1}$

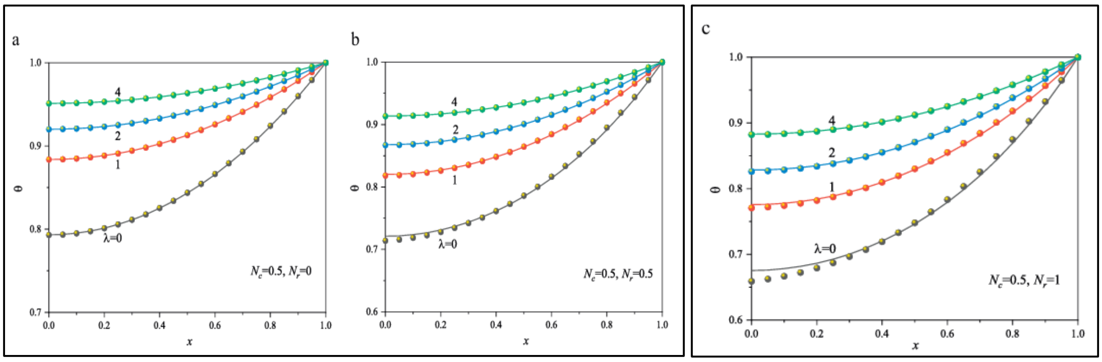

Fig. 4 Temperature distribution for $\boldsymbol{N_c = 0.5}$ and $\boldsymbol{N_r = 0, 0.5, 1}$ obtained by [11]

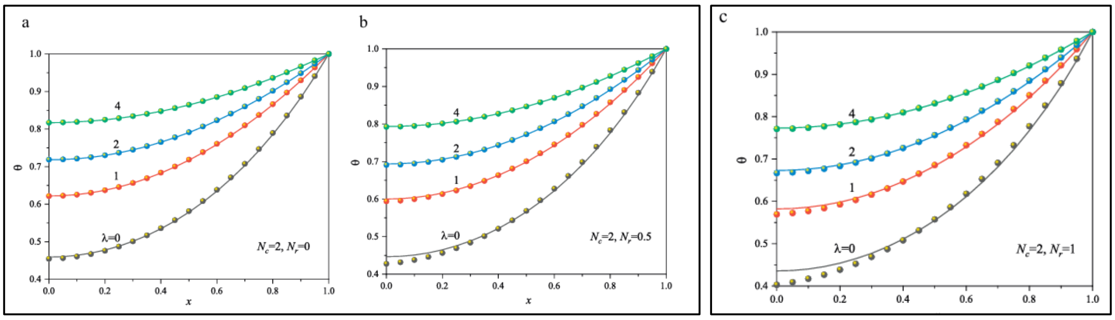

Fig. 5 Temperature distribution for $\boldsymbol{N_c = 2}$ and $\boldsymbol{N_r = 0, 0.5, 1}$ obtained by [11]

Table 1 Temperature at $\boldsymbol{x = 0}$ for $\boldsymbol{N_c = 0.5}$

	$\boldsymbol{\theta(0)}$			
N_r	$\lambda = 0$	$\lambda = 1$	$\lambda = 2$	$\lambda = 4$
0	0.793278	0.883699	0.920065	0.951072
0.5	0.721046	0.819916	0.867552	0.913454
1	0.675478	0.675478	0.828264	0.882839

Table 2 Temperature at $\boldsymbol{x = 0}$ for $\boldsymbol{N_c = 2.0}$

	$\theta(0)$			
N_r	$\lambda = 0$	$\lambda = 1$	$\lambda = 2$	$\lambda = 4$
0	0.459098	0.621443	0.717926	0.816861
0.5	0.446524	0.599839	0.693271	0.793212

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1	0.435646	0.581721	0.672556	0.772848

Table 3 presents the comparison of the tip temperature obtained by present method which is shooting method and semi-analytical approach based on Taylor series expansion by [11] in order to validate the accuracy of shooting method for $N_c = 1.0$ with $N_r = 1,2,3$ and $\lambda = 1,2,3$. It can be observed that the result obtained using shooting method are in close agreement with those reported in [11] for all parameter values. The result show that the fin tip temperature decreases when N_r increases, indicating enhanced heat loss due to radiation effect. When the value of λ increase the fin tip temperature increase, as higher thermal conductivity enhances heat conduction along the fin. These prove that the shooting method are accurate and reliable because these trends are consistent with physical expectations.

Table 3 Comparison of the tip temperature obtained by present method and [11] for $\boldsymbol{N_c = 1.0}$

	$\boldsymbol{\theta(0)}$					
	Shooting Method			**Semi-analytical approach based on Taylor series expansion [11]**		
N_r	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$
1.0	0.7062	0.7741	0.8164	0.6987	0.7708	0.8165
2.0	0.6576	0.7257	0.7705	0.6396	0.7170	0.7261
3.0	0.6222	0.6894	0.7352	0.5917	0.6739	0.7261

Table 4 and Table 5 presents the output from MATLAB for iteration count and the CPU times for all parameter values used in this study. The output shows that the shooting method demonstrates good convergence behaviour. The solution converges within 1 to 6 iterations for $N_c = 0.5$, for $N_c = 2$, the iteration within 2 to 6 iterations while $N_c = 1.0$, the iteration within 5 to 6 iterations. This shows that only a small number of iterations is required to obtain accurate solution, even when the problem is nonlinear.

In addition, the CPU times proves the computational efficiency of the shooting method. The output shows that, most simulations require 0.01 seconds and all are cases are completed in under 0.15 seconds. Even the highest value which is 0.1307 still very fast. Nevertheless, the computation time remains minimal. Thus, these results confirm that the shooting method is both stable and computationally efficient for solving nonlinear boundary value problems in heat transfer.

Table 4 Number of Iteration and CPU Time for $\boldsymbol{N_c = 0.5}$

$\boldsymbol{N_r}$	$\boldsymbol{\lambda}$	**Iteration**	**CPU(s)**
0	0	1	0.1307
	1	4	0.0366
	2	4	0.0172
	4	4	0.0150
0.5	0	5	0.0300
	1	3	0.0134
	2	4	0.0174
	4	4	0.0109
1.0	0	6	0.0262
	1	5	0.0178
	2	4	0.0169
	4	5	0.0183

Table 5 Number of Iteration and CPU Time for $\boldsymbol{N_c = 2.0}$

N_r	λ	Iteration	CPU(s)
0	0	2	0.0357
	1	4	0.0202
	2	5	0.0160
	4	4	0.0149
0.5	0	6	0.0501
	1	5	0.0303
	2	5	0.0203
	4	4	0.0102
1.0	0	5	0.0206
	1	6	0.0220
	2	5	0.0153
	4	4	0.0111

Table 6 Number of Iteration and CPU Time for $\boldsymbol{N_c = 1.0}$

N_r	λ	Iteration	CPU(s)
1	1	5	0.0447
	2	6	0.0089
	3	5	0.0061
2	1	5	0.0101
	2	5	0.0084
	3	5	0.0059
3	1	5	0.0072
	2	6	0.0088
	3	5	0.0079

4. Conclusion

Numerical solutions were obtained for various values of the convection parameter N_c, radiation parameter N_r and thermal conductivity parameter λ. Shooting method successfully determines the unknown initial condition and generates physically accurate temperature distribution along the fin. The results obtained for temperature distribution both graphically and numerically, match those from previous study. The close agreement between the present results and those reported in the previous literature confirms that the shooting method were accurate and reliable for solving the nonlinear BVP. In terms of computational efficiency, the shooting method shows rapid convergences with few iterations and low computational time even for a nonlinear problem. In conclusion, the shooting method is an accurate and efficient numerical approach for solving nonlinear heat transfer problems. However, the present work is limited to steady-state, one-dimensional analysis. Future work may extend the analysis to higher-dimensional problems and compare the method with other numerical techniques.

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Conflict of Interest

Authors declare that there is no conflict of interests regarding the publication of the paper.

Author Contribution

The authors confirm contribution to the paper as follows: ***study conception and design:*** *Norzidah Mad Ainal;* ***solve the governing equation:*** *Norzidah Mad Ainal;* ***analysis and interpretation of results:*** *Norzidah Mad Ainal, Fazlina Aman;* ***draft manuscript preparation:*** *Norzidah Mad Ainal. All authors reviewed the results and approved the final version of the manuscript.*

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