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Numerical Study of Eigenvalue Problem Using Iterative Methods

Fatin Nur Azrin Ayob[1], Fazlina Aman[1*]

[1] *Department of Mathematics and Statistics, Faculty of Applied Sciences and Technology, UTHM Kampus Cawangan Pagoh, Hab Pendidikan Tinggi Pagoh, KM 1, Jalan Panchor, 84600 Pagoh, Muar, Johor, MALAYSIA.*

*Corresponding Author: fazlina@uthm.edu.my

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Abstract

This study focuses on solving generalized eigenvalue problems by iterative method which are power method and Rayleigh quotient iteration. The primary purpose is to compare the performance of these two methods with respect to convergence speed, accuracy and computational efficiency when applied to a workforce transition model among three sectors which are information technology (IT), education, and media. The transition matrix used is a symmetric, stochastic with dominant eigenvalue corresponds to the steady-state distribution of workers. The power method algorithm is used to approximate the largest eigenvalue and corresponding eigenvector through repeated matrix-vector multiplication and normalization, while the Rayleigh quotient iteration used inverse iteration with dynamic shifting for better convergence. Rayleigh quotient iteration achieves faster convergence with two to three iterations than the power method that up to 250 iterations. Both methods yield the same result with a dominant eigenvalue of 1.0027 and consistent steady-state eigenvectors. The results are verified with eigen function which comes in MATLAB.

1. Introduction

Eigenvalue is a fundamental concept in linear algebra that plays a crucial role in various mathematical and scientific applications such as solving systems of linear differential equations and analyzing stability in dynamical systems. The eigenvalue problem can be expressed by the equation $A\mathbf{v} = \lambda\mathbf{v}$ for a square matrix A, and is a fundamental concept in numerical linear algebra with extensive applications across engineering and scientific computing [1]. The solutions are crucial in domains like the stability analysis of dynamical systems [2], vibrational mode analysis in structural mechanics [3] and steady-state analysis of Markov processes in socio-economic modelling [4]. In practical applications such as predictive workforce analysis, these problems often involve large or structured matrices. For example, [4] modeled occupational transitions using a column-stochastic matrix, with the dominant eigenvector defining the long-term equilibrium distribution. Therefore, accurate and efficient computation on this eigenpair is essential for reliable forecasting.

Direct solution methods like the QR algorithm have a computational complexity of $O(n^3)$. This makes them infeasible for high dimensional system due to prohibitive memory and processing demands [5]. Therefore, iterative methods have become the preferred approach for approximating extremal eigenvalues particularly when exploiting matrix eigenvalues subset of the spectrum is needed [6]. The power method is the simplest iterative algorithm for finding dominant eigenvalues [7]. While the algorithm is simple and memory efficient, but the convergence is linear with a rate dependent on the ratio $|\lambda 2/\lambda 1|$ and closely spaced spectra can cause slow convergence [8]. To overcome this limitation, more sophisticated iterative techniques have developed. The Rayleigh quotient iteration is more powerful extension of the inverse power method that incorporates an adaptive shift, updated using the Rayleigh quotient at each step [9]. This shifting strategy significantly accelerates convergence, often achieving cubic convergence rates near an eigenvalue, where the number of correct digits approximately triples with each iteration [10]. Although Rayleigh quotient iteration is more computation per iteration due to the required solution of a shifted linear system, the substantial reduction in iteration count frequently leads to greater overall efficiency, particularly for well-conditioned problems.

This study has three objectives. First, to develop a comprehensive understanding of the mathematical background behind generalized eigenvalue problems. Second, to conduct a comparative analysis of various iterative algorithms, including the power method and Rayleigh quotient iteration, in solving the eigenvalue problems. Lastly, the study aims to evaluate the convergence behaviour and computational efficiency of each of these methods, providing insights into their practical applications. The problem studied in this paper is a discrete-time occupational transition model of workers who flow between the IT, education and media sectors with fixed size each year. The model emphasizes that over time, workers are distributed based on annual transition probabilities between sectors. Such transition mobilities, like the transition matrix, are symmetric and stochastic, yielding an eigenvalue problem that describes the systems stationary state behaviour based on the predictive model described by [4], this research assumes that the total number of workers remains the same at any point in time.

2. Research Methodology

The problem investigated is a discrete-time occupational transition model which are workers circulating among the sectors of information technology (IT), education and media, each having a fixed number of people each year. The model focuses on the long-term distribution of workers based on annual transition probabilities from one sector to another. These transition dynamics, like the transition matrix, which is symmetric and stochastic, lead to an eigenvalue problem that governs the system's steady-state behaviour. Based on the predictive model described by [4] this research assumes that the total number of workers remains the same at any point in time. Annual transition rates are fixed and uniform, as are the labour force forms throughout the year. In fact, there is no external cause to either increase or decrease employment levels. Initial labour distribution is known, and transitions occur uniformly at year-end. Since the transition matrix is real and symmetric, it has real eigenvalues and orthogonal eigenvectors. Furthermore, the dominant eigenvalue equals unity conveying that number of employees is conserved with corresponding eigenvector representing this equilibrium distribution. Based on these assumptions, the discrete-time evolution of the occupational distribution is governed by the following eigenvalue problem [4]:

$$v_{n+1} = Av_{n+1}, \tag{1}$$

where:

- $v = \begin{bmatrix} v_{IT} \\ v_{Education} \\ v_{Media} \end{bmatrix}$ is the sector population vector at year n,
- A is the 3×3 column stochastic transition matrix.
- n: Number of years for prediction

Thus, the transition matrix A is defined as:

$$A = \begin{bmatrix} 0.97 & 0.02 & 0.01 \\ 0.02 & 0.97 & 0.01 \\ 0.01 & 0.01 & 0.987 \end{bmatrix} \tag{2}$$

Each element A_{ij} defined as the proportion of workers moving from sector *j* to sector *i* per year.

2.1 Power Method

Step 1: Choose an initial guess nonzero vector $v^{(0)}$.
Let:

$$v^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{3}$$

Step 2: Iteration for $k = 0,1,2 \dots$

$$v_{k+1} = Av_k \tag{4}$$

After each multiplication, normalize the newly obtained vector v_{k+1} to prevent numerical instability and maintain manageable size of the vectors. This is done by dividing the vector by its norm:

$$v_{k+1} = \frac{v_{k+1}}{\|v_{k+1}\|} \tag{5}$$

Once we have the normalized eigenvector, the dominant eigenvalue is estimated using the Rayleigh quotient:

$$\lambda_{k+1} = \frac{{v_{k+1}}^T A v_{k+1}}{{v_{k+1}}^T v_{k+1}} \tag{6}$$

The Rayleigh quotient is used solely to estimate the eigenvalue corresponding to the dominant eigenvector and does not involve Rayleigh Quotient Iteration.

Step 3: The change in the eigenvalue is less than the tolerance

$$|\lambda_1 - \lambda_0| < \mathcal{E} \tag{7}$$

2.2 Rayleigh Quotient Iteration

Step 1: Choose an initial guess nonzero vector $\boldsymbol{v}^{(\mathbf{0})}$ and let it satisfy (1)

Step 2: Initial eigenvalue estimation λ_0

$$\lambda_0 = \frac{{v_0}^T A v_0}{{v_0}^T v_0} \tag{8}$$

Step 3: Iteration for $k = 0,1,2 \dots$

$$\lambda_{k+1} = \frac{{v_k}^T A v_k}{{v_k}^T v_k} \tag{9}$$

Compute the eigenvalue estimate

Solve the shifted linear system $(A - \lambda_1 I)v_1 = v_0$. Solving $(A - \lambda_1 I)v_1 = v_0$ for v_1 involves solving the linear system which can be done using Gaussian elimination or LU decomposition. Then normalize v_1 to avoid numerical instability and use (5).

Step 4: The iterations continue until convergence

3. Results and Discussion

The results are obtained using power method and Rayleigh quotient iteration for the eigenvalue problem associated with workforce transitions analysis on IT, Education, and Media sectors. This involves analysing transition matrices that attempt to predict worker movements between these industries in the next decade. The power method and Rayleigh quotient iteration are two iterative methods for computing dominant eigenvalues and associated eigenvector. The convergence behaviour between the two methods is compared and the numerical data and graphical illustrations are used to support the analysis and demonstrate the effect of the parameters on the solution's accuracy and convergence speed. Lastly, the results from the two methods are validated and compared their performance on solving workforce transition problem.

3.1 Power Method

The power method iteration results from Table 1 suggest slow convergence of both the eigenvalue and eigenvectors by 250 iterations. At the beginning, eigenvalue is 1.0024 and it gradually converges to 1.0027 meaning that the system turns into steady state as the iterations progress. Likewise, the three dimensions have different starting eigenvectors, v_1, v_2 and v_3 as shown in the Table 1 and becomes stable after a few iterations.

At the 250th iteration, the eigenvalue and eigenvectors have reach stable values. The dominant eigenvalue is 1.0027 with the corresponding eigenvectors:

$$v_1 = 0.5257$$

$$v_2 = 0.5257$$

$$v_3 = 0.6687$$

These findings indicate that the division of labour among IT, education and media sectors has become stabilized. Behaviour of eigenvalue and eigenvectors at the end show that this method works well to approximate the dominant eigenvalue and its corresponding steady-state eigenvector.

Table 1 Power method iteration table for the problem

Iteration	Eigenvalue	v_1	v_2	v_3
1	1.0024	0.576	0.576	0.58004
2	1.0024	0.57469	0.57469	0.58264
3	1.0024	0.5734	0.5734	0.58517
4	1.0024	0.57215	0.57215	0.58762
5	1.0024	0.57092	0.57092	0.59
⋮	⋮	⋮	⋮	⋮
250	1.0027	0.52575	0.52575	0.66871

The plot in Fig. 1 depicts the rate of convergence of the largest eigenvalue using the power method over 250 iterations. At first the eigenvalue increases more quickly, corresponding to the fast approach of the power method toward the actual eigenvalue. It does slow down as iterations approach infinity and the graph do flatten out showing that the eigenvalue has settled at around 1.0027 now. This behaviour is also consistent with power method, in which the eigenvalue converges fast at the early iterations and flattens toward to its final value after sufficient number of steps. The plot presents the method's efficiency in finding the dominant eigenvalue, with diminishing changes after a certain point, showing that the process reaches a steady state after sufficient iterations.

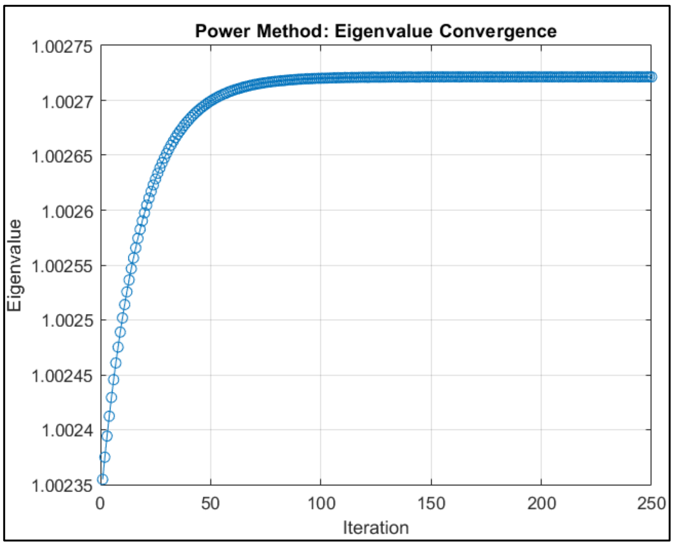

Fig. 1 Eigenvalue convergence obtained using the power method

3.2 Rayleigh Quotient Iteration

From Table 2 Rayleigh quotient iteration approach shows a quick convergence of the dominant eigenvalue and its corresponding eigenvector. On the second iteration, the eigenvalue is 1.0027 and this value corresponds with the result derived from the power method, which verifies that it is being done correctly. Eigenvectors for workers in the IT, education and media industries also converge rapidly. At iteration two the eigenvector values settle at approximately:

$$\boldsymbol{v_1} = \mathbf{0.5257}$$

$$\boldsymbol{v_2} = \mathbf{0.5257}$$

$$\boldsymbol{v_3} = \mathbf{0.6687}$$

This rate of convergence, which is faster than the power method, reflects greater speed of performance for the Rayleigh quotient iteration approach and requires less iterations to obtain the results. The solutions obtained by Rayleigh quotient iteration are an accurate and fast estimate of the dominant eigenvalue and its associated eigenvector. Lastly, for this class of problem, Rayleigh quotient iteration is more efficient.

Table 2 Rayleigh quotient iteration table for the problem

Iteration	Eigenvalue	v_1	v_2	v_3
1	1.0027	0.52575	0.52575	0.66871
2	1.0027	0.52575	0.52571	0.66878

Fig. 2 shows the strong convergence of the eigenvalue by the Rayleigh quotient iteration algorithm. Eigenvalue starts at 1.0027, but it steadily increases with iteration until value becomes 1.0027214627 occurring during second iteration. This indicates the eigenvalue converges very fast, with just two iterations it gets a stable value. As opposed to the power method, which often needs an increased number of iterations to converge, the Rayleigh quotient iteration provides a quicker convergence and it is more accurate than power method for calculation of dominant eigenvalue. The plot illustrates that the Rayleigh quotient approach is iteratively improving the eigenvalue estimate between the first two iterations and it is immediately clear show how fast the correct value.

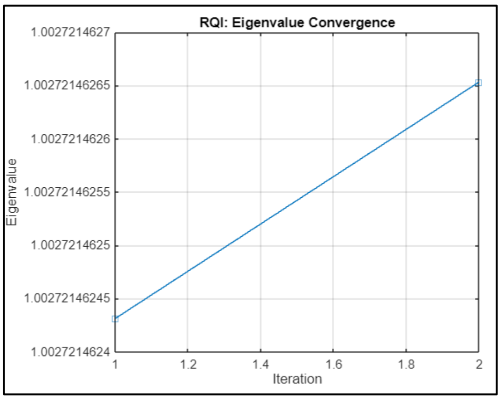

Fig. 2 Eigenvalue convergence obtained using the Rayleigh quotient iteration

3.3 Convergence Comparison of Two Method

Table 3 shows the comparison between two methods which are power method and Rayleigh quotient iteration.

Table 3 Comparison of power method and Rayleigh quotient iteration

Criteria	Power method	Rayleigh quotient iteration
Convergence speed	Slower convergence and requires many iterations to stabilize	Faster convergence and stabilizes quickly within a few iterations
Number of iterations to converge	Requires 250 iterations to reach a stable eigenvalue	Achieves convergence within 2-3 iterations
The accuracy	Accurate, but may take longer for high precision	High accuracy and achieved quickly

Rayleigh quotient iteration algorithm is faster and more efficient than power method, in terms of the rate of convergence and computational cost. Rayleigh quotient iteration converges rapidly to the leading eigenvalue with a smaller number of iterations and therefore is more favourable for the computation in terms of time and machine resources. In contrast, while the power method is still applicable, it typically requires more iterations to compute an accurate answer, making it less efficient for problems that require quick solutions

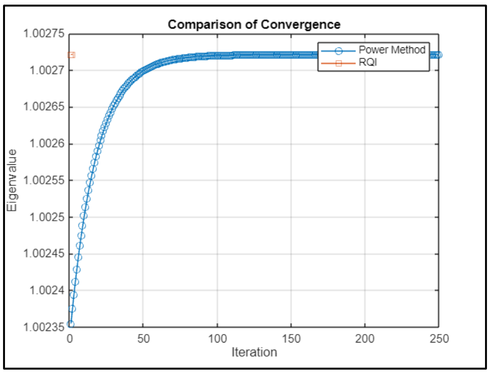

Fig. 3 Graph comparison of power method and Rayleigh quotient iteration

3.4 Validation

The outputs from the power method and Rayleigh quotient iteration methods are aligned closely with the provided output, verifying these computations. Both methods lead to the same dominant eigenvalue of 1.0027, which validates the estimated eigenvalue. The eigenvectors we find $v_1 = 0.5257$,$v_2 = 0.5257$ and $v_1$0.6687 agree with the proportionality in the system although slightly different normalized to the printed output $v_1 = 0.5257$,$v_2 = 0.5257$ and $v_1 = 0.6688$. The population projections of IT, education, and media after 10 years are close, which validates the results. The results obtained by both approaches are therefore in accordance with the target output and have been validated.

Eigenvalue output from MATLAB:

- $\lambda_{IT} = 0.9500$
- $\lambda_{Education} = 0.9743$
- $\lambda_{Media} = 1.0027$

Table 4 Eigenvectors output from MATLAB table

IT	Education	Media
0.7071	0.4729	0.5257
-0.7071	0.4729	0.5257
0.0000	-0.7435	0.6688

4. Conclusion

This study is dedicated to computing eigenvalue problems by applying two iterative methods which are the power method and Rayleigh quotient iteration. The objective of this research is to see how these methods can be used to solve the eigenvalue problems such as a workforce transition model where workers transit across three sectors, which are IT, education and media within a span of 10 years. The model uses a transition matrix to represent the annual probabilities of workers shifting between these sectors, and the objective is to predict the long-term distribution of workers across these industries. By calculating the dominant eigenvalue and associated eigenvector, the study attempts to interpret what the steady-state distribution of workers will look like after several years. Furthermore, the work compares the convergence speed, computational efficiency and accuracy of power method and Rayleigh quotient iteration in solving the eigenvalue problem. This comparison provides information on the application of these iterative methods in practice to provide input to the choice of an appropriate method for a given problem and computational requirement. The key findings from applying the power method and Rayleigh quotient iteration on this study shows that Rayleigh quotient iteration converged significantly faster than the power method, particularly when the eigenvalues of the transition matrix were close to each other. While the power method was effective in identifying the dominant eigenvalue, it took more iterations and showed less precision, especially with closely spaced eigenvalues. Rayleigh quotient iteration provided more accurate results, refining both the eigenvalue and eigenvector estimates with fewer iterations. The choice of initial eigenvector and tolerance levels affected the convergence speed for both methods, but Rayleigh quotient iteration remained more efficient and accurate overall. Graphs comparing the two methods showed that Rayleigh quotient iteration reached a steady eigenvalue much quicker than the power method.

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Conflict of Interest

Authors declare that there is no conflict of interest regarding the publication of the paper.

Author Contribution

The authors confirm contribution to the paper as follows: ***study conception and design:*** *Fatin Nur Azrin Ayob;* ***solve eigenvalue problem:*** *Fatin Nur Azrin Ayob;* ***analysis and interpretation of results:*** *Fatin Nur Azrin Ayob;* ***draft manuscript preparation:*** *Fatin Nur Azrin Ayob, Fazlina Aman. All authors reviewed the results and approved the final version of the manuscript.*

References

[1] Golub, G. H., & van der Vorst, H. A. (2000). Eigenvalue computation in the 20th century. Journal of Computational and Applied Mathematics, 123(1), 35–65. https://doi.org/10.1016/S0377-0427(00)00413-1

[2] Bisoffi, A., Beerens, R., Heemels, W. P. M. H., Nijmeijer, H., van de Wouw, N., & Zaccarian, L. (2020). To stick or to slip: A reset PID control perspective on positioning systems with friction. Annual Reviews in Control, 49, 37–63. https://doi.org/10.1016/j.arcontrol.2020.04.010

[3] Wang, B., An, H., Xie, H., & Mo, Z. (2025). A new subspace iteration algorithm for solving generalized eigenvalue problems in vibration analysis. Journal of Computational and Applied Mathematics, 468, 116622. https://doi.org/10.1016/j.cam.2025.116622

[4] Jiang, Y. (2022). Study on eigenvalue and eigenvector introduction. Journal of Physics: Conference Series, 2282(1), 012004. https://doi.org/10.1088/1742-6596/2282/1/012004

[5] Bai, Z.-Z. (2025). On greedy randomized coordinate updating iteration methods for solving symmetric eigenvalue problems. Applied Numerical Mathematics, 216, 76–97. https://doi.org/10.1016/j.apnum.2025.04.010

[6] El Hachimi, A., Jbilou, K., & Ratnani, A. (2025). Krylov subspace methods for large multidimensional eigenvalue computation. Applied Numerical Mathematics, 208, 205–221. https://doi.org/10.1016/j.apnum.2024.01.017

[7] Yang, K., Xia, L., & Tay, Y. C. (2023). Dominant Eigenvalue-Eigenvector Pair Estimation via Graph Infection (No. arXiv:2208.00982). arXiv. https://doi.org/10.48550/arXiv.2208.00982

[8] NotesOnPowerMethod.pdf. (n.d.). Retrieved June 10, 2025, from https://www.cs.utexas.edu/~flame/Notes/NotesOnPowerMethod.pdf

[9] Prinn, A. G., & Habets, E. A. P. (2025). Acoustic eigenvalue estimation using Rayleigh quotient iteration with a sparse matrix pencil. Applied Acoustics, 232, 110573. https://doi.org/10.1016/j.apacoust.2025.110573

[10] Jia, Z. (2009, June 12). On Convergence of the Inexact Rayleigh Quotient Iteration with MINRES. arXiv.Org. https://arxiv.org/abs/0906.2238v3