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Validation of SAWI Transformation for Nonlinear Differential Equations Using Fourth-Order Runge-Kutta

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Abstract

This paper investigates the method of Shifted and Weighted Integral (SAWI) transformation for solving a nonlinear system of ordinary differential equations (ODEs) with quadratic nonlinearity. The approach semi-analytical solution is validated by comparison with the classical fourth-order Runge-Kutta (RK4) method. Numerical results show that the absolute errors demonstrating strong agreement between both approaches. The SAWI technique transforms the connected nonlinear system into a set of differential equations in the sigma-domain. The agreement between the two approaches for the quadratically nonlinear system confirms the reliability and efficiency of the SAWI transformation as a semi-analytical technique, providing analytically valid and computationally efficient analysis of such systems.

1. Introduction

In engineering, physics, biology, and economics, ordinary differential equations (ODEs) are fundamental for modeling systems that change over time [1]. Traditional numerical techniques such as Runge-Kutta Fourth Order (RK4) method are commonly used [2]. However, the RK4 method provides limited analytical insight and can be computationally expensive for stiff systems [3]. Nonlinear systems introduce further complexity, including bifurcations, sensitivity to initial conditions, and chaos [4]. Semi-analytical methods like the SAWI Transformation have been developed to offer a compromise between computational performance and analytical tractability [5]. The SAWI transformation converts differential equations into algebraic forms in the sigma-domain, preserving the system's time-dependent structure while simplifying the solution process [6]. Although the SAWI transformation has demonstrated effectiveness in linear and selected nonlinear problems, its applicability to couple nonlinear systems has not been extensively explored. This limitation motivates the present study. Such systems are prevalent in real-world models across mechanical vibrations, population dynamics, and chemical kinetics [7].

The objective of this research is to solve nonlinear ODE systems using SAWI Transformation and test accuracy and efficiency by comparing results with the RK4 method. This research focuses on first order coupled system with nonlinear equation to perform a systematic evaluation of the method's applicability, solution accuracy, and computational efficiency. By integrating analytical and numerical techniques, this study shows that SAWI is a practical semi-analytical approach for nonlinear ODEs and provides dependable numerical results.

2. Methodology

2.1 Nonlinear System of ODEs

A nonlinear system of first-order ODEs is expressed as:

$$\begin{cases} \frac{dy_1}{dx} = f_1(x, y_1, y_2, \ldots, y_n) \\ \frac{dy_2}{dx} = f_2(x, y_1, y_2, \ldots, y_n) \\ \quad\vdots \\ \frac{dy_n}{dx} = f_n(x, y_1, y_2, \ldots, y_n) \end{cases} \tag{1}$$

where $\boldsymbol{f_i}$ are nonlinear functions of the dependent variables. In this study, the following system is examined:

System 1:

$$\begin{cases} y'_1 = y_1 \\ y'_2 = {y_1}^2 - y_2 \end{cases} \tag{2}$$

with initial conditions

$$y_1(0) = 1, y_2(0) = 0 \tag{3}$$

This system features quadratic nonlinearities and is solved using the SAWI transformation, with results validated against RK4 solutions.

2.2 SAWI Transformation

2.2.1 Definition of SAWI transformation

The SAWI transforms of the function $\omega(t), y \geq 0$ is given by [8]
k denotes the shifting parameter introduce to enhance the flexibility of the SAWI Transformation

$$S\{\omega(t)\} = \frac{1}{\sigma^2}\int_0^\infty \omega(t)e^{-\left(\frac{1}{\sigma}\right)t}\,dt = T(\sigma), \sigma > 0 \tag{4}$$

$$\begin{aligned} Se^{kt}w(kt) &= \frac{1}{\sigma^2}\int_0^\infty \omega(t)e^{-\left(\frac{1-k\sigma}{\sigma}\right)t}\,dt = T\left(\frac{\sigma}{1-k\sigma}\right) \\ &= \frac{1}{(1-k\sigma)^2}\left[\frac{(1-k\sigma)^2}{\sigma^2}\int_0^\infty \omega(t)e^{-\left(\frac{1-k\sigma}{\sigma}\right)t}\,dt\right] \\ &= \frac{1}{(1-k\sigma)^2}T\left(\frac{\sigma}{1-k\sigma}\right) \end{aligned} \tag{5}$$

2.2.2 SAWI transformation of derivatives of function

Table 1 presents the SAWI transformation of derivatives of function. [8]

Table 1 SAWI transformation of derivatives of function

No.	Function of derivatives	SAWI transformation of derivatives
1.	$\omega'(t)$	$\frac{1}{\sigma}T(\sigma) - \frac{1}{\sigma^2}\omega(0)$
2.	$\omega''(t)$	$\frac{1}{\sigma^2}T(\sigma) - \frac{1}{\sigma^3}\omega(0) - \frac{1}{\sigma^2}\omega'(0)$
3.	$\vdots$	$\vdots$
4.	$\omega^{(\rho)}(t)$	$\frac{1}{\sigma^\rho}T(\sigma) - \sum_{k=0}^{\rho-1}\left(\frac{1}{\sigma}\right)^{\rho-(k-1)}\omega^{(k)}(0)$

2.2.3 Characteristics of SAWI transformation

The following are the characteristics of SAWI Transformation, which make it suitable for solving system of ODEs.

i. Linearity property of SAWI transformation

$$S[\alpha\omega_1(t) + \beta\omega_2(t)] = [\alpha T_1(\sigma) + \beta T_2(\sigma)]$$

where α and β are arbitrary constants.

ii. Scaling property of SAWI transformation

$$S\{\omega(kt)\} = kT(k\sigma)$$

iii. Translation property of SAWI transformation

$$S\{e^{kt}\omega(kt)\} = \frac{1}{(1-k\sigma)^2}T\left(\frac{\sigma}{1-k\sigma}\right)$$

2.2.4 Some fundamental functions with their SAWI transformation

Some fundamental concepts of SAWI transformation involve its basic formula for transforming functions into algebraic expressions and the derivative formula, which allows differential terms to be simplified while automatically incorporating initial conditions. [8]

Table 2 Fundamental functions with their SAWI transformation

No.	$\boldsymbol{\omega(t)}$	$\boldsymbol{S\{\omega(kt)\} = kT(k\sigma)}$
1.	1	$\frac{1}{\sigma}$
2.	t	1
3.	t^2	2σ
4.	$t^2, \rho \in N$	$\rho!\,\sigma^{\rho-1}$
5.	$t^\rho, \rho > -1$	$T(\rho+1)\sigma^{\rho-1}$
6.	e^{at}	$\frac{1}{\sigma(1-a\sigma)}$
7.	$\sin(at)$	$\frac{a}{1+(a\sigma)^2}$
8.	$\cos(at)$	$\frac{1}{\sigma(1+(a\sigma)^2)}$
9.	$\sinh(at)$	$\frac{a}{1-(a\sigma)^2}$
10.	$\cosh(at)$	$\frac{1}{\sigma(1-(a\sigma)^2)}$

Symbols	Meaning / Description
t	Time variable
$\omega(t)$	Original function in the t-domain
k	Scaling constant (usually a positive constant)
$S\{\cdot\}$	SAWI transformation operator
σ	Variable in the SAWI domain
$T(\cdot)$	Gamma function
ρ	real constant

$\mathbb{N}$	Set of natural numbers (positive integers)
a	Real (or complex) constant
e^{at}	Exponential function
$sin(at), cos(at)$	Trigonometric functions
$sinh(at), cosh(at)$	Hyperbolic functions

2.2.5 Inverse SAWI transformation of some fundamental functions

The function $\omega(t)$ is called the inverse SAWI transformation of the function $T(\sigma)$ if it has the following property $S\{\omega(t)\} = T(\sigma)$. In symbolic form, it is written as $S^{-1}\{T(\sigma)\} = \omega(\sigma)$. [8]

Table 3 Inverse SAWI transformation of some fundamental functions

No.	$\boldsymbol{T(\sigma)}$	$\boldsymbol{S\{\omega(kt)\} = kT(k\sigma)}$
1.	$\frac{1}{\sigma}$	1
2.	1	t
3.	σ	$\frac{t^2}{2!}$
4.	$\sigma^{\rho-1}, \rho \in N$	$\frac{t^\rho}{\rho!}$
5.	$\sigma^{\rho-1}, \rho > -1$	$\frac{t^\rho}{T(\rho+1)}$
6.	$\frac{1}{\sigma(1-a\sigma)}$	e^{at}
7.	$\frac{a}{1+(a\sigma)^2}$	$\frac{\sin(at)}{a}$
8.	$\frac{1}{\sigma(1+(a\sigma)^2)}$	$\cos(at)$
9.	$\frac{a}{1-(a\sigma)^2}$	$\frac{\sinh(at)}{a}$
10.	$\frac{1}{\sigma(1-(a\sigma)^2)}$	$\cosh(at)$

2.3 Runge-Kutta Fourth-order Method

The RK4 method is used as a numerical benchmark [5]. For an ODE, $\boldsymbol{y' = f(t, y)}$ the solution advances from $\boldsymbol{t_n}$ to $\boldsymbol{t_{n+1} = t_n + h}$:

$$\begin{aligned}
k_1 &= hf(t_n, y_n) \\
k_2 &= hf(t_n + h/2, y_n + k_1/2), \\
k_3 &= hf(t_n + h/2, y_n + k_2/2), \\
k_4 &= hf(t_n + h, y_n + k_3), \\
y_{n+1} &= y_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).
\end{aligned} \tag{6}$$

For systems, this process is applied simultaneously to each equation. RK4 provides fourth-order accuracy and serves as a reliable standard for comparing the SAWI-derived solutions.

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3. Result and Discussion

3.1 Example 1

Consider system 1 in (2) with initial conditions (3):

3.1.1 SAWI Transformation Solution

Applying SAWI transformation:

$$\begin{cases} S[y'_1] = S[y_1] \\ S[y'_2] = S[-y_2] + S[y_1{}^2] \end{cases} \tag{7}$$

Using the SAWI derivative property:

$$\begin{cases} \frac{1}{\sigma}Y_1(\sigma) - \frac{1}{\sigma^2}y_1(0) = Y_1(\sigma) \\ \frac{1}{\sigma}Y_2(\sigma) - \frac{1}{\sigma^2}y_2(0) = -Y_2(\sigma) + W(\sigma) \end{cases}$$

Then for the first equation, substitute the initial condition

$$\begin{gathered} \frac{1}{\sigma}Y_1 - \frac{1}{\sigma^2}(1) = Y_1 \\ \frac{1}{\sigma}Y_1 - Y_1 = \frac{1}{\sigma^2} \\ \left(\frac{1}{\sigma} - 1\right)Y_1 = \frac{1}{\sigma^2} \\ Y_1(\sigma) = \frac{1}{\sigma(1-\sigma)} \end{gathered}$$

Applying the inverse SAWI:

$$y_1(\sigma) = e^t \tag{8}$$

Then for the second equation, substitute the initial condition

$$\begin{gathered} \frac{1}{\sigma}Y_2 - (0) = -Y_2 + W \\ \frac{1}{\sigma}Y_2 + Y_2 = W \\ \left(\frac{1}{\sigma} + 1\right)Y_2 = W \\ Y_2 = \frac{\sigma}{\sigma+1}W \end{gathered} \tag{9}$$

From (6), we know $\boldsymbol{W(\sigma) = S[y_1{}^2]}$

Since $\boldsymbol{y_1(t) = e^t}$, so $\boldsymbol{y_1{}^2 = e^{2t}}$

According to table 2,

$$S[e^{at}] = \frac{1}{\sigma(1-a\sigma)}$$

For a= **2**:

$$S[e^{2t}] = \frac{1}{\sigma(1-2\sigma)}$$

So

$$W(\sigma) = \frac{1}{\sigma(1-2\sigma)}$$

Substitute W into (8)

$$Y_2(\sigma) = \frac{\sigma}{\sigma+1}\left(\frac{1}{\sigma(1-2\sigma)}\right)$$

$$Y_2(\sigma) = \frac{1}{(\sigma+1)(1-2\sigma)}$$

Using partial fractions:

$$\frac{1}{(\sigma+1)(1-2\sigma)} = \frac{A}{1-2\sigma} + \frac{B}{\sigma+1}$$
$$1 = A(\sigma+1) + B(1-2\sigma)$$
$$1 = (A-2B)\sigma + (A+B)$$

Let A=2B, $B = \frac{1}{3}, A = \frac{2}{3}$

$$Y_2(\sigma) = \frac{2}{3(1-2\sigma)} + \frac{1}{3(\sigma+1)}$$
$$Y_2(\sigma) = \frac{2}{3}.\frac{1}{(1-2\sigma)} + \frac{1}{3}.\frac{1}{\sigma+1} \tag{10}$$

Apply SAWI to $\boldsymbol{\frac{e^{at}-1}{a}}$

$$S\{\omega(t)\} = \frac{1}{\sigma^2}\int_0^\infty \omega(t)e^{-\left(\frac{1}{\sigma}\right)t}\,dt$$
$$S\left\{\frac{e^{at}-1}{a}\right\} = \frac{1}{\sigma^2}\int_0^\infty \frac{e^{at}-1}{a}e^{-\left(\frac{1}{\sigma}\right)t}\,dt$$
$$= \frac{1}{a\sigma^2}\left[\int_0^\infty e^{at}e^{-\left(\frac{1}{\sigma}\right)t}\,dt - \int_0^\infty e^{-\left(\frac{1}{\sigma}\right)t}\,dt\right]$$
$$= \frac{1}{a\sigma^2}\left[\int_0^\infty e^{-\left(\frac{1}{\sigma}-a\right)t}\,dt - \int_0^\infty e^{-\left(\frac{1}{\sigma}\right)t}\,dt\right]$$

Both integrals are exponential integrals:

$$\int_0^\infty e^{-kt}\,dt = \frac{1}{k}\ , \qquad k > 0$$

So

$$= \frac{1}{a\sigma^2}\left[\frac{1}{\frac{1}{\sigma}-a} - \frac{1}{\frac{1}{\sigma}}\right]$$
$$= \frac{1}{a\sigma^2}\left[\frac{\sigma}{1-a\sigma} - \sigma\right]$$
$$= \frac{1}{a\sigma}\left[\frac{1}{1-a\sigma} - 1\right]$$
$$= \frac{1}{a\sigma} \times \frac{a\sigma}{1-a\sigma}$$
$$= \frac{1}{1-a\sigma}$$

The SAWI transform of $\frac{e^{at}-1}{a}$ is $\frac{1}{1-a\sigma}$, hence its inverse transform returns $\frac{e^{at}-1}{a}$.

$$\boldsymbol{S^{-1}\left[\frac{1}{(1-a\sigma)}\right] = \frac{e^{at}-1}{a}}$$

Applying inverse SAWI to $\boldsymbol{y_2(\sigma)}$

$$\boldsymbol{y_2(t) = \frac{1}{3}(e^{2t}-1) + \frac{1}{3}(1-e^{-t})}$$

$$\boldsymbol{y_2(t) = \frac{1}{3}e^{2t} + \frac{1}{3}e^{-t}}$$

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3.1.2 Final SAWI Solution

$$y_1(t) = e^t$$
$$y_2(t) = \frac{1}{3}e^{2t} + \frac{1}{3}e^{-t}$$

3.2 Comparison with RK4 Method

The SAWI solutions for $\boldsymbol{y_1(t)}$ and $\boldsymbol{y_2(t)}$ were computed analytically using the transformation method. For comparison, the RK4 method was implemented numerically in MATLAB with a fixed step size h=0.01 to obtain the approximate solutions at selected time points. The absolute errors between the SAWI and RK4 results are presented in Table 4.

Table 4 Comparison of SAWI and RK4 solutions for Example 1

t	SAWI, y_1	RK4, y_1	Error, y_1	SAWI, y_2	RK4, y_2	Error, y_2
0.0	1.0000	1.0000	0.00e+00	0.0000	0.0000	0.00e+00
0.1	1.1051	1.1051	8.47e-08	0.1055	0.1055	3.36e-07
0.2	1.2214	1.2214	1.87e-07	0.2243	0.2243	7.02e-07
0.3	1.3498	1.3498	3.11e-07	0.3604	0.3604	1.11e-06
0.4	1.4918	1.4918	4.58e-07	0.5184	0.5184	1.55e-06
0.5	1.6487	1.6487	6.32e-07	0.7039	0.7039	2.05e-06
0.6	1.8221	1.8221	8.38e-07	0.9237	0.9237	2.61e-06

As shown in Table 4, the numerical results obtained from the RK4 method (implemented in MATLAB) closely match the SAWI-derived solutions with errors remaining within the order of $\boldsymbol{10^{-6}}$ over the given time interval confirming the accuracy of the SAWI transformation.

3.3 Graphical Representation of Results for SAWI and RK4

Fig. 1 and Fig. 2 show the comparison results of $\boldsymbol{y_1(t)}$ and $\boldsymbol{y_2(t)}$ with the RK4

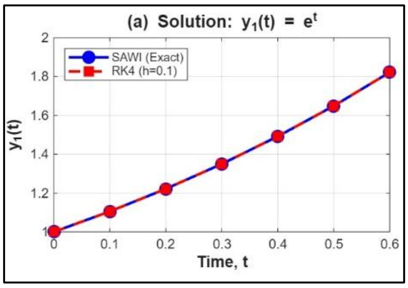

Fig. 1 Comparison of $\boldsymbol{y_1}$ obtained using the SAWI transformation with the RK4

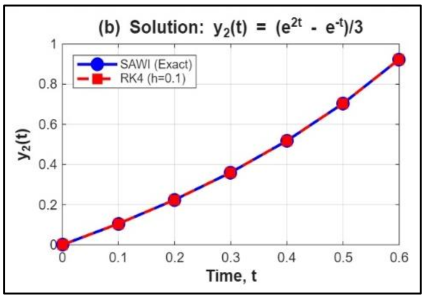

Fig. 2 Comparison of $\boldsymbol{y_2}$ obtained using the SAWI transformation with RK4

The SAWI transformation was successfully applied to solve the nonlinear system. The analytical solutions derived are:

$$y_1(t) = e^t, y_2(t) = \frac{1}{3}e^{2t} + \frac{1}{3}e^{-t}. \tag{10}$$

The solutions were compared with those obtained by the RK4 method. The absolute errors between SAWI and RK4 were within the order of 10^{-6} to 10^{-7} throughout the period $t \in [0,0.6]$. The graph illustrates the similarity between the SAWI and the RK4 curves nearly indistinguishable, as a comparison (Fig. 1, and Fig. 2), which provides graphical verification of the SAWI method.

Results show the SAWI Transformation can effectively deal with quadratic nonlinearities in a systematic way by using the SAWI operator, explicitly applying the initial conditions as derivative transformation rules, solving algebraic equations in domain- σ, and applying an inverse transformation to obtain results in the time domain. This semi-analytical approach makes explicit the formulas of the functions which helps in gaining a better understanding on how dynamics of systems work. For example, y_1 has exponential growth and y_2, both growing and decay modes in y_2. These dynamic characteristics are not easily observed from numerical value alone.

If there are known pairs of transformations of the nonlinear variable made available, the approach is beneficial. Otherwise, it might be used only for simple nonlinearities. Current applications still require some manual algebraic manipulation, suggesting a need for future development of automated symbolic calculation tools.

Nevertheless, SAWI approach does provide a compromise of information that is as good as it gets between analysis transparency and exactitude of numbers. This research has already proven helpful for replacing traditional numerical methods like RK4, in applications demanding precise numerical solutions.

4. Conclusion

This work has been able to prove that the SAWI Transformation approach can be used and work in the solution of nonlinear systems of ODEs. This approach has been successfully used for nonlinear systems and can also generate an accurate semi-analytical solution. Numerical testing against the RK4 method confirmed the solution's accuracy, with errors between 10^{-6} and 10^{-7}, graphical tests of the solution have proven excellent results.

SAWI technique offers several advantages. The semi-analytical framework that combines analytical and numerical methods provides clear and useful solutions that are mathematically clear. The initial conditions are directly included; this makes the process of resolving it simpler. However, it is recognized that this work is limited to a specialty area of nonlinearities (quadratic or exponential), first-order systems, and manual algebraic manipulation.

This study is limited to first-order coupled nonlinear ODEs with quadratic nonlinearity. The proposed SAWI transformation is demonstrated using a single test problem with a known analytical behavior, and results were compared with the fourth-order Runge–Kutta (RK4) method. The solution process also required manual algebraic steps, which could make it less practical for more complicated nonlinear systems.

In future research, the SAWI transformation could be applied to higher-order and multi-dimensional nonlinear systems, including stiff and chaotic models. Further investigation on convergence, stability, and error estimation is also recommended. Additionally, creating automated symbolic or computational tools to implement the method would greatly improve its usefulness in solving real-world scientific and engineering problems.

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Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Author Contribution

The authors confirm contribution to the paper as follows: ***study conception and design, analysis, draft manuscript preparation:*** *Ahmad Naeem Siddiqi Ibrahim;* ***supervision, manuscript review and editing****: Noor Azliza Abd Latif, Norziha Che Him. All authors reviewed the results and approved the final version of the manuscript.*

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